Secondary Two Mathematics: An Integrated Approach
Module 3
Quadratic Equations

By

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In partnership with the
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Module 3 – Quadratic Equations

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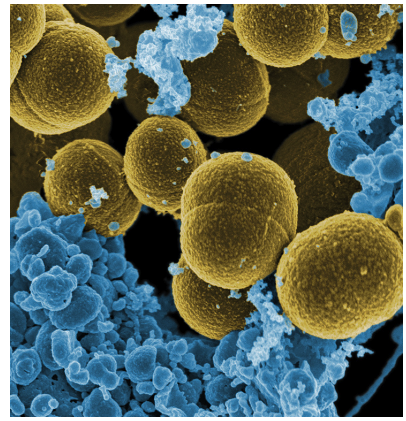
3.1 Experimenting with Exponents
A Develop Understanding Task



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Travis and Miriam are studying bacterial growth. They were surprised to find that the population of the bacteria doubled every hour.

1. Complete the following table and plot the data on the graph at the end of this task.

Hours into the study	0	1	2	3	4
Bacteria population (in thousands)	4				

2. Write an equation for P, the population of the bacteria, as a function of time, t, and verify that it produces correct populations for t = 1, 2, 3, and 4 hours.

Travis and Miriam want to create a table with more entries; specifically, they want to fill in the population at each half hour. Unfortunately, they forgot to make these measurements so they decide to estimate the values.

Travis makes the following claim:

> "If the population doubles in 1 hour, then half that growth occurs in the first half-hour and the other half in the second half-hour. So for example, we can find the population at t = ½ by finding the average of the populations at t = 0 and t = 1."

3. Fill in the parts of the table below that you've already computed, and then decide how you might use Travis' strategy to fill in the missing data. Also plot Travis' data on the graph at the end of the task.

Hours into the study	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
Bacteria population (in thousands)	4								

4. Comment on Travis' idea. How does it compare to the table generated in problem 1? For what kind of function would this reasoning work?

Miriam suggests they should fill in the data in the table in the following way:

> "To make the estimates, I noticed that the population increases by the same factor each hour, and I think that this property should hold over each half-hour interval as well."

4. Fill in the parts of the table below that you've already computed in problem 1, and then decide how you might use Miriam's new strategy to fill in the missing data. As in the table in problem 1, each entry should be multiplied by some constant factor in order to produce consistent results. Use this constant multiplier to complete the table. Also plot Miriam's data on the graph at the end of this task.

Hours into the study	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
Bacteria population (in thousands)	4								

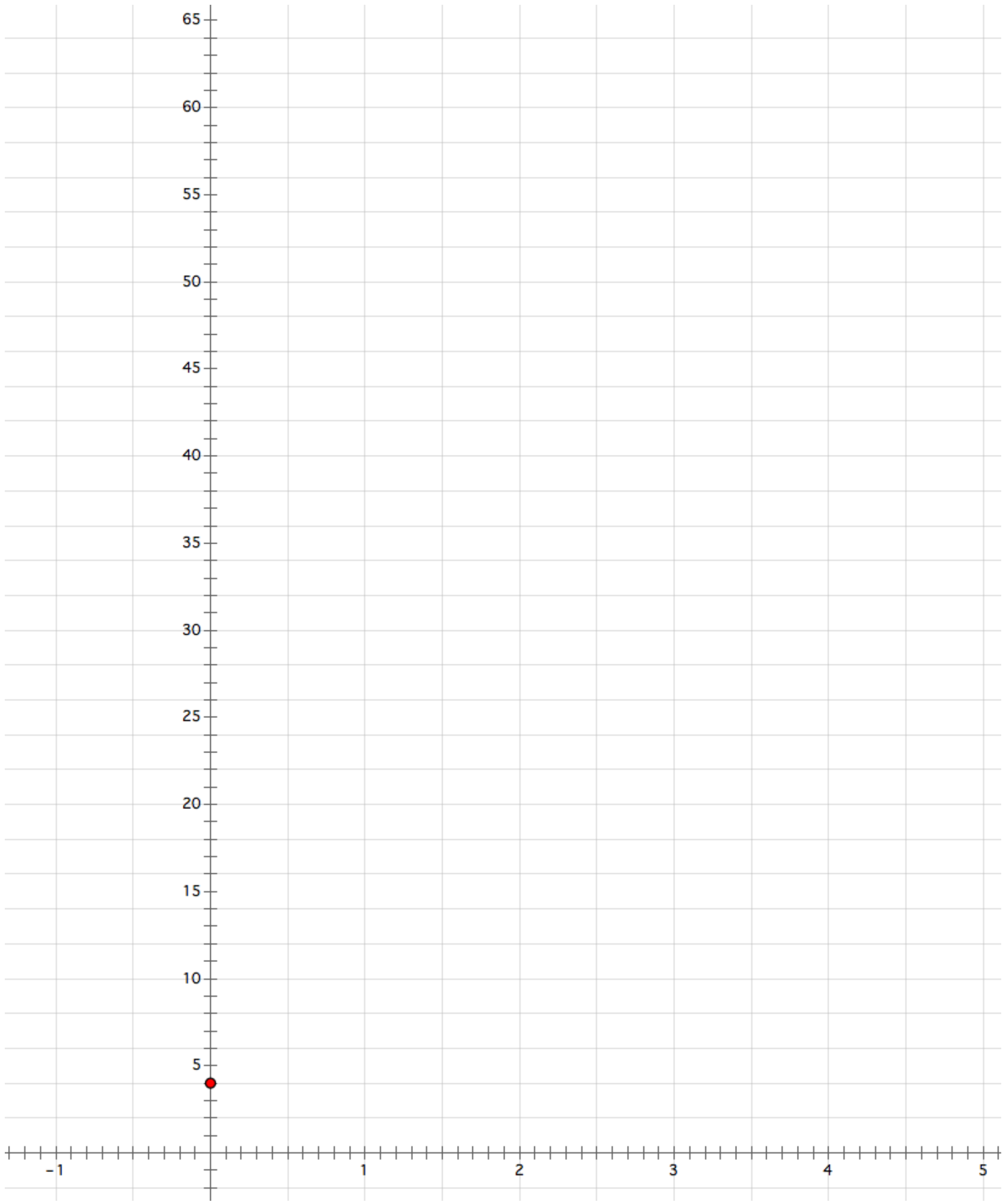
5. What if Miriam wanted to estimate the population every 20 minutes instead of every 30 minutes? What multiplier would she use for every third of an hour to be consistent with the population doubling every hour? Use this multiplier to complete the following table.

Hours into the study	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2	$\frac{7}{3}$	$\frac{8}{3}$	3
Bacteria population (in thousands)	4									

6. What number did you use as a multiplier to complete the table in problem 4?

7. What number did you use as a multiplier to complete the table in problem 5?

8. Give a detailed description of how you would estimate the population, P, at $t = \frac{5}{3}$ hours.


65
60
55
50
45
40
35
30
25
20
15
10
5
-1
1
2
3
4
5

3.1 Experimenting with Exponents – Teacher Notes

A Develop Understanding Task

Purpose: This task surfaces the idea that data exists on the intervals between the whole number increments of a continuously increasing exponential function. Students will consider potential strategies for calculating this data at equal fractional increments of time so that the multiplicative pattern inherent in exponential functions is maintained. Students who are familiar with the work of such tasks as *Geometric Meanies* from the MVP Secondary I curriculum may choose to use a radical, such as $\sqrt{2}$ as the factor to multiply each entry in the table by to get the next entry when the data is spaced in $\frac{1}{2}$ units increments, and $\sqrt[3]{2}$ when the data is spaced in $\frac{1}{3}$ unit increments. The task provides students with an opportunity to connect these multipliers with the exponents that represent the increments of time in the exponential function, pointing toward the definition for rational exponents, $b^{1/n} = \sqrt[n]{b}$.

Core Standards Focus:

N.RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.

Related Standards: A.SSE.1

Launch (Whole Class):
As part of the launch, read the context and ask students to complete the table and write an equation to fit the context and the data in the table. This work should be easy for students to do, since they are familiar with using exponential equations like $P = 4(2)^t$ to represent exponential growth over whole number increments of time. Present the new situation—Travis and Miriam need data at smaller increments of time than one hour increments—and set students to work to consider Travis' and Miriam's proposed methods for producing this data.

Explore (Small Group):
Listen for students who can argue that Travis' strategy implies that the points at the intervals in-between the whole number data points are being approximated by straight line segments , rather than by the smooth curve suggested by the data already recorded. Miriam's approach acknowledges that the curve is produced by multiplying each entry in the table by a constant factor (2, in the case of the whole number increments) to get the next data entry—a multiplicative recursive approach. Consequently, students need to find a factor that can be used over and over again to produce the data at half-hour or third-hour increments. Students may guess and check such a factor (we know it has to be larger than 1 since the bacteria population is growing, but less

than 2 since the population can't double in less than an hour). Allow students to use a guess and check strategy, if that approach surfaces. Other students may recall similar work with geometric means from the task *Geometric Meanies* in the MVP Secondary I curriculum. This might lead them to propose $\sqrt{2}$ and $\sqrt[3]{2}$ as the common ratios for the half-hour and third-hour increments, respectively. Since rational exponents have not been introduced to students it is not anticipated that this idea will come up in the exploration. If it does, select students to present their reasoning during the whole class discussion.

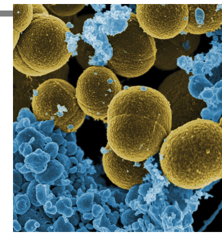
Discuss (Whole Class):

Focus the discussion on the table of data for the half-hour increments. Ask students what they think about Travis' strategy. Press students to acknowledge that since the data is obviously not linear based on the hourly data, we would not suspect it to be linear on smaller subintervals of an hour. Ask students to predict when most of the bacterial growth might occur during the first hour—during the first half of the hour or during the second half of the hour—and justify their reasons for thinking so. Listen for arguments that suggest that the bacterial growth is constantly increasing, suggesting that as time passes the growth should be greater when examining equal intervals of time.

Move on to Miriam's approach and have students present the factor they used as the constant multiplier. If available in the student work, have a student present first who used a guess and test strategy, perhaps closing in on a decimal number such as 1.412 as a factor that produces half-hour data that is consistent with the hourly data. Then select a student to present who used $\sqrt{2}$ as the factor. Press for a justification such as, "We need a number that, when multiplied by itself, gives us 2; which fits the definition of a square root." Next, turn students' attention to the graph and their equation, and asked if anybody tried to plot a point on the graph where $x = ½$ by substituting ½ into the exponential equation for x? Point out that the calculator yields a result that is consistent with the value obtained when multiplying by the square root of 2, that is, $4 \cdot \sqrt{2} = 4 \cdot 2^{1/2}$ as far as the calculator is concerned.

Depending on your class you might want to end the discussion at this point, or continue to pursue this new type of exponent. The next task, *Half Interested*, continues to pursue this idea of using a fraction as an exponent. You might choose to continue this discussion at this point in time if there are students who can argue that $2^{1/2}$ can reasonably be defined as $\sqrt{2}$ since by properties of exponents $2^{1/2} \cdot 2^{1/2} = 2^1 = 2$ in the same way that $\sqrt{2} \cdot \sqrt{2} = 2$. You can return to this idea as part of discussion of the next task if students have not yet surfaced this idea in their thinking.

Aligned Ready, Set, Go: Solving Quadratic and Other Equations 3.1



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Ready, Set, Go!

Ready

Topic: Comparing additive and multiplicative patterns.

The sequences below exemplify either an additive (arithmetic) or a multiplicative (geometric) pattern. Identify the type of sequence, fill in the missing values on the table and write an equation.

1.

Term	1st	2nd	3rd	4th	5th	6th	7th	8th
Value	2	4	8	16	32	a.	b.	c.

d. Type of Sequence: e. Equation:

2.

Term	1st	2nd	3rd	4th	5th	6th	7th	8th
Value	66	50	34	18	a.	b.	c.	d.

e. Type of Sequence: f. Equation:

3.

Term	1st	2nd	3rd	4th	5th	6th	7th	8th
Value	-3	9	-27	81	a.	b.	c.	d.

e. Type of Sequence: f. Equation:

4.

Term	1st	2nd	3rd	4th	5th	6th	7th	8th
Value	160	80	40	20	a.	b.	c.	d.

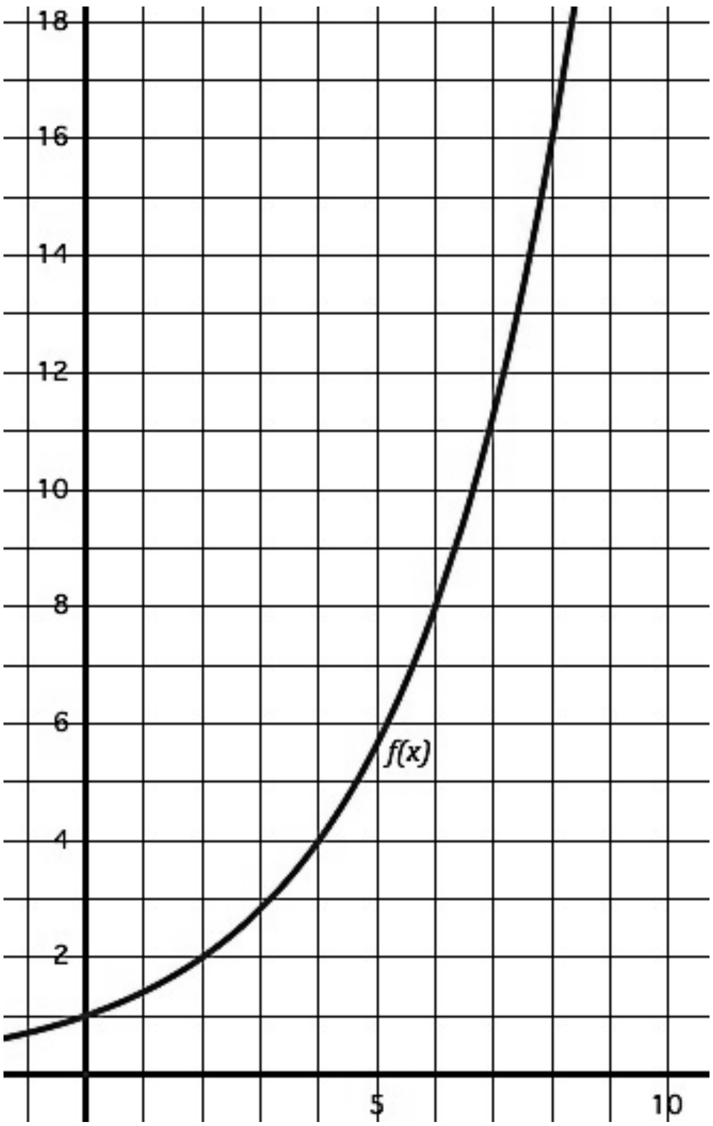
e. Type of Sequence: f. Equation:

5.

Term	1st	2nd	3rd	4th	5th	6th	7th	8th
Value	-9	-2	5	12	a.	b.	c.	d.

e. Type of Sequence: f. Equation:

Use the graph of the function to find the desired values of the function. Also create an explicit equation for the function.



6. Find the value of *f(2)*

7. Find where *f(x) = 4*

8. Find the value of *f(6)*

9. Find where *f(x) = 16*

10. What do you notice about the way that inputs and outputs for this function relate? (Create an in-out table if you need to.)

11. What is the explicit equation for this function?

Set

Topic: Evaluate the expressions with rational exponents.

Fill in the missing values of the table based on the growth that is described.

12. The growth in the table is triple at each whole year.

Years	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
bacteria	2		6						

13. The growth in the table is triple at each whole year.

Years	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2	$\frac{7}{3}$	$\frac{8}{3}$
bacteria	2			6					

14. The values in the table grow by a factor of four at each whole year.

Years	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
bacteria	2		8						

Go

Topic: Simplifying exponents

Simplify the following expressions using exponent rules and relationships, write your answers in exponential form. (For example: $2^2 \cdot 2^5 = 2^7$)

15. $3^2 \cdot 3^5$

16. $\frac{5^3}{5^2}$

17. 2^{-5}

18. 17^0

19. $\frac{7^5}{7^2} \cdot \frac{7^3}{7^4}$

20. $\frac{3^{-2} \cdot 3^5}{3^7}$

3.2 Half Interested

A Solidify Understanding Task

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Carlos and Clarita, the Martinez twins, have run a summer business every year for the past five years. Their first business, a neighborhood lemonade stand, earned a small profit that their father insisted they deposit in a savings account at the local bank. When the Martinez family moved a few months later, the twins decided to leave the money in the bank where it has been earning 5% interest annually. Carlos was reminded of the money when he found the annual bank statement they had received in the mail.

"Remember how Dad said we could withdraw this money from the bank when we are twenty years old," Carlos said to Clarita. "We have $382.88 in the account now. I wonder how much that will be five years from now?"

1. Given the facts listed above, how can the twins figure out how much the account will be worth five years from now when they are twenty years old? Describe your strategy and calculate the account balance.

2. Carlos calculates the value of the account one year at a time. He has just finished calculating the value of the account for the first four years. Describe how he can find the next year's balance, and record that value in the table.

year	amount
0	382.88
1	402.02
2	421.12
3	443.23
4	465.39
5	

3. Clarita thinks Carlos is silly calculating the value of the account one year at a time, and says that he could have written a formula for the n^{th} year and then evaluated his formula when $n = 5$. Write Clarita's formula for the n^{th} year and use it to find the account balance at the end of year 5.

4. Carlos was surprised that Clarita's formula gave the same account balance as his year-by-year strategy. Explain, in a way that would convince Carlos, why this is so.

"I can't remember how much money we earned that summer," said Carlos. "I wonder if we can figure out how much we deposited in the account five years ago, knowing the account balance now?"

5. Carlos continued to use his strategy to extend his table year-by-year back five years. Explain what you think Carlos is doing to find his table values one year at a time, and continue filling in the table until you get to -5, which Carlos uses to represent "five years ago".

year	amount
-5	
-4	
-3	
-2	
-1	364.65
0	382.88
1	402.02
2	421.12
3	443.23
4	465.39
5	

6. Clarita evaluated her formula for $n = -5$. Again Carlos is surprised that they get the same results. Explain why Clarita's method works.

Clarita doesn't think leaving the money in the bank for another five years is such a great idea, and suggests that they invest the money in their next summer business, *Curbside Rivalry* (which, for now, they are keeping top secret from everyone, including their friends). "We'll have some start up costs, and this will pay for them without having to withdraw money from our other accounts."

Carlos remarked, "But we'll be withdrawing our money halfway through the year. Do you think we'll lose out on this year's interest?"

"No, they'll pay us a half-year portion of our interest," replied Clarita.

"But how much will that be?" asked Carlos.

7. Calculate the account balance and how much interest you think Carlos and Clarita should be paid if they withdraw their money ½ year from now. Remember that they currently have - $382.88 in the account, and that they earn 5% annually. Describe your strategy.

Carlos used the following strategy: He calculated how much interest they should be paid for a full year, found half of that, and added that amount to the current account balance.

Clarita used this strategy: She substituted ½ for *n* in the formula $A = 382.88(1.05)^n$ and recorded this as the account balance.

8. This time Carlos and Clarita didn't get the same result. Whose method do you agree with and why?

Clarita is trying to convince Carlos that her method is correct. "Exponential rules are multiplicative, not additive. Look back at your table. We will earn $82.51 in interest during the next four years. If your method works we should be able to take half of that amount, add it to the amount we have now, and get the account balance we should have in two years, but it isn't the same."

9. Carry out the computations that Clarita suggested and compare the result for year 2 using this strategy as opposed to the strategy Carlos originally used to fill out the table.

10. The points from Carlos' table (see question 2) have been plotted on the graph at the end of this task, along with Clarita's function. Plot the value you calculated in question 9 on this same graph. What does the graph reveal about the differences in Carlos' two strategies?

11. Now plot Clarita's and Carlos' values for ½ year (see question 8) on this same graph.

"Your data point seems to fit the shape of the graph better than mine," Carlos conceded, "but I don't understand how we can use ½ as an exponent. How does that find the correct factor we need to multiply by? In your formula, writing $(1.05)^5$ means multiply by 1.05 five times, and writing $(1.05)^{-5}$ means divide by 1.05 five times, but what does $(1.05)^{1/2}$ mean?"

Clarita wasn't quite sure how to answer Carlos' question, but she had some questions of her own. She decided to jot them down, including Carlos' question:

- What numerical amount do we multiply by when we use $(1.05)^{1/2}$ as a factor?
- What happens if we multiply by $(1.05)^{1/2}$ and then multiply the result by $(1.05)^{1/2}$ again? Shouldn't that be a full year's worth of interest? Is it?
- If multiplying by $(1.05)^{1/2} \cdot (1.05)^{1/2}$ is the same as multiplying by 1.05, what does that suggest about the value of $(1.05)^{1/2}$?

12. Answer each of Clarita's questions listed above as best as you can.

As Carlos is reflecting on this work, Clarita notices the date on the bank statement that started this whole conversation. "This bank statement is three months old!" she exclaims. "That means the bank will owe us ¾ of a year's interest."

"So how much interest will the bank owe us then?", asked Carlos.

13. Find as many ways as you can to answer Carlos' question: How much will their account be worth in ¾ of a year (nine months) if it earns 5% annually and is currently worth $382.88?


465
460
455
450
445
440
435
430
425
420
415
410
405
400
395
390
385
380
(4, 465.39)
(3, 443.23)
(2, 421.12)
(1, 402.02)
(0, 382.88)
Selected: 6

3.2 Half Interested – Teacher Notes

A Solidify Understanding Task

Purpose: In the context of predicting the account balance at different times for an account earning 5% interest annually, students examine the role of positive and negative integer exponents as well as the need for rational exponents. Tables, graphs and reasoning based on the definition of radicals and rules of exponents are used to attach meaning to using fractions such as ½ or ¾ as exponents.

Core Standards Focus:

N.RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.

Related Standards: A.SSE.1

Launch (Whole Class):
Students who used the MVP Secondary I curriculum have previously met Carlos and Clarita as industrious teenage twins who create various summer business ventures to earn money (see the *Pet Sitters* context from Secondary I, module 1). Read through the context at the beginning of the task, making sure students are aware that we are working with an exponential context that has both a future and a past. This will allow positive, negative and rational exponents to make sense in the context. Point out that Carlos is using a recursive approach to extend the exponential growth, while Clarita is using an explicit function. In Secondary I students learned that a recursive approach was appropriate for discrete data, but not for continuous data. This issue will surface for Carlos as he tries to reason through the data that exists between the whole number increments of time. Point out to students that the work in this task is similar to work in the previous task, and then set them to work on the different elements of the story.

Explore (Small Group):
Students should make good progress on these problems through question 10, since the work and reasoning is very similar to the work in the previous task, *Experimenting with Exponents.* Give appropriate support, as needed, particularly to the idea that a negative integer exponent implies dividing by a factor of 1.05 as many times as represented by the magnitude of the exponent.

If the discussion in the previous task surfaced the idea of using a fraction as an exponent—and its meaning as a radical—then the rest of the task should confirm students thinking. If the previous discussion did not make this connection, then this task should do so now. Listen for students making sense of Clarita's questions and identify students who can relate the idea that

$(1.05)^{1/2} \cdot (1.05)^{1/2} = (1.05)^1 = 1.05$ is consistent with the additive properties of exponents, and implies that $(1.05)^{1/2} = \sqrt{1.05}$ based on the definition of square root.

Discuss (Whole Class):
Start the whole class discussion by focusing on question 12, students' answers to Clarita's questions. Make sure the conversation solidifies the reasonableness of using a fraction, such as $1/n$, to represent an n^{th} root. Discuss the third-of-an-hour table in the previous task to add additional support to this claim.

Next, discuss question 13, (how to find the value of the account after ¾ of a year has elapsed), along with question 8 from the previous task, (how would you find the bacterial population at $5/3$ hours). Help students recognize that, based on the properties of exponents, $2^{1/3} \cdot 2^{1/3} \cdot 2^{1/3} \cdot 2^{1/3} \cdot 2^{1/3}$ can be written as $2^{5/3}$ and gives the same result as $\left(\sqrt[3]{2}\right)^5$, or that $\left(\sqrt[4]{1.05}\right)^3$ gives the same result as $1.05^{3/4}$. Allow students to calculate these values using technology and show that the results are consistent with the population and interest contexts provided in the tasks.

Aligned Ready, Set, Go: Solving Quadratic and Other Equations 3.2

Name: Solving Quadratic and Other Equations 3.2

Ready, Set, Go!

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Ready

Topic: Simplifying Radicals

A very common radical expression is a square root. One way to think of a square root is the number that will multiply by itself to create a desired value. For example: $\sqrt{2}$ *is the number that will multiply by itself to equal 2. And in like manner* $\sqrt{16}$ *is the number that will multiply by itself to equal 16, in this case the value is 4 because 4 x 4 = 16. (When the square root of a square number is taken you get a nice whole number value. Otherwise an irrational number is produced.)*

This same pattern holds true for other radicals such as cube roots and fourth roots and so forth. For example: $\sqrt[3]{8}$ *is the number that will multiply by itself three times to equal 8. In this case it is equal to the value of 2 because* $2^3 = 2 \times 2 \times 2 = 8$.

With this in mind radicals can be simplified. See the examples below.

Example 1: Simplify $\sqrt{20}$ $\sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{2 \cdot 2 \cdot 5} = 2\sqrt{5}$	*Example 2*: Simplify $\sqrt[5]{96}$ $\sqrt[5]{96} = \sqrt[5]{2^5 \cdot 3} = 2\sqrt[5]{3}$

Simplify each of the radicals.

1. $\sqrt{40}$
2. $\sqrt{50}$
3. $\sqrt[3]{16}$
4. $\sqrt{72}$
5. $\sqrt[4]{81}$
6. $\sqrt{32}$
7. $\sqrt[5]{160}$
8. $\sqrt{45}$
9. $\sqrt[3]{54}$

Set

Topic: Finding arithmetic and geometric means and making meaning of rational exponents.

You may have found arithmetic and geometric means in your prior work. Finding arithmetic and geometric means requires finding values of a sequence between given values from non-consecutive terms. In each of the sequences below determine the means and show how you found them.

Find the *arithmetic* means for the following. Show your work.

10.

x	1	2	3
y	5		11

11.

x	1	2	3	4	5
y	18	a.	b.	c.	-10

12.

x	1	2	3	4	5	6	7
y	12	a.	b.	c.	d.	e.	-6

Find the *geometric* means for the following. Show your work.

13.

x	1	2	3
y	3		12

14.

x	1	2	3	4
y	7	a.	b.	875

15.

x	1	2	3	4	5	6
y	4	a.	b.	c.	d.	972

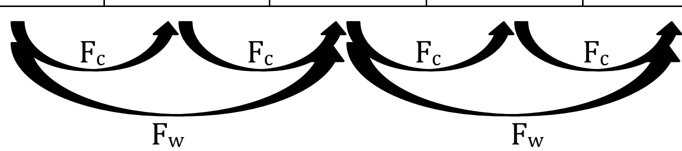
Fill in the tables of values and find the factor used to move between whole number values, F_w, as well as the factor, F_c, used to move between each column of the table.

16.

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	4	a.	16	b.	c.

d. $F_w =$

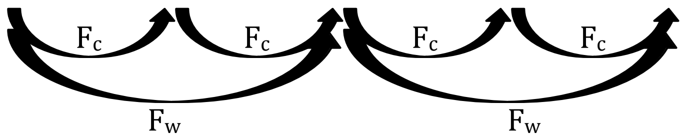
e. $F_c =$



17.

x	0	½	1	$\frac{3}{2}$	2
y	4	a.	8	b.	c.

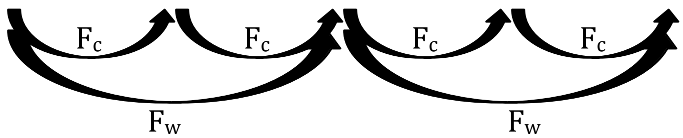
d. $F_w =$
e. $F_c =$



18.

x	0	½	1	$\frac{3}{2}$	2
y	5	a.	15	b.	c.

d. $F_w =$
e. $F_c =$



Go

Topic: Evaluating functions

Find the desired values for each function below.

19. $f(x) = 2x - 7$

a. Find $f(-3)$

b. Find $f(x) = 21$

c. Find $f\left(\frac{1}{2}\right)$

20. $g(x) = 3^x(2)$

a. Find $g(-4)$

b. Find $g(x) = 162$

c. Find $g\left(\frac{1}{2}\right)$

21. $I(t) = 210(1.08^t)$

a. Find $I(12)$

b. Find $I(t) = 420$

c. Find $I\left(\frac{1}{2}\right)$

22. $h(x) = x^2 + x - 6$

a. Find $h(-5)$

b. Find $h(x) = 0$

c. Find $h\left(\frac{1}{2}\right)$

23. $k(x) = -5x + 9$

a. Find $k(-7)$

b. Find $k(x) = 0$

c. Find $k\left(\frac{1}{2}\right)$

24. $m(x) = (5^x)2$

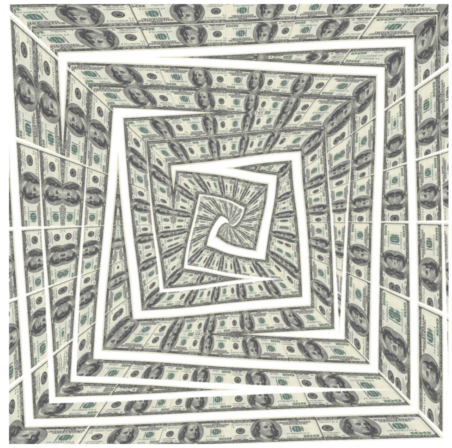
a. Find $m(-2)$

b. Find $m(x) = 1$

c. Find $m\left(\frac{1}{2}\right)$

3.3 More Interesting!

A Solidify Understanding Task



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Carlos now knows he can calculate the amount of interest earned on an account in smaller increments than one full year. He would like to determine how much money is in an account each month that earns 5% annually with an initial deposit of $300.

He starts by considering the amount in the account each month during the first year. He knows that by the end of the year the account balance should be $315, since it increases 5% during the year.

1. Complete the table showing what amount is in the account each month during the first twelve months.

deposit											1 year
$300											$315

2. What number did you multiply the account by each month to get the next month's balance?

Carlos knows the exponential equation that gives the account balance for this account on an annual basis is $A = 300(1.05)^t$

Based on his work finding the account balance each month, Carlos writes the following equation for the same account: $A = 300(1.05^{1/12})^{12t}$.

3. Verify that both equations give the same results. Using the properties of exponents, explain why these two equations are equivalent.

4. What is the meaning of the $12t$ in this equation?

Carlos shows his equation to Clarita. She suggests his equation could also be approximated by $A = 300(1.004)^{12t}$, since $(1.05)^{1/12} \approx 1.004$. Carlos replies, "I know the 1.05 in the equation $A = 300(1.05)^t$ means I am earning 5% interest annually, but what does the 1.004 mean in this equation?"

5. Answer Carlos' question. What does the 1.004 mean in $A = 300(1.004)^{12t}$?

The properties of exponents can be used to explain why $[(1.05)^{1/12}]^{12t} = 1.05^t$. Here are some more examples of using the properties of exponents with rational exponents. For each of the following, simplify the expression using the properties of exponents, and explain what the expression means in terms of the context.

6. $(1.05)^{1/12} \cdot (1.05)^{1/12} \cdot (1.05)^{1/12}$

7. $[(1.05)^{1/12}]^6$

8. $(1.05)^{-1/12}$

9. $(1.05)^2 \cdot (1.05)^{1/4}$

10. $\dfrac{(1.05)^2}{(1.05)^{1/2}}$

Carlos and Clarita now have two equations representing the balance of their 5% account after *t* years: $A = 300(1.05)^t$ and $A = 300(1.05^{1/12})^{12t}$. In both of these equations *t* represents the amount of time the money has been in the account in terms of *years*.

Carlos and Clarita know they can use their equations for fractions of a year by expressing *t* in terms of a portion of a year, for example, using *t = 2.5* for two and one-half years or $t = 1/12$ for one month. They are wondering if they can write an equation that would find the account balance in terms of *t* months or *t* days.

11. Write an equation that will find the account balance in terms of *t* months.

12. Write an equation that will find the account balance in terms of *t* days.

13. The account balance is currently $382.88. Write an equation that will find the account balance *t* months ago.

3.3 More Interesting – Teacher Notes

A Solidify Understanding Task

Purpose: The purpose of this task is to verify the properties of exponents that students know for integer exponents also work for rational exponents. In the context of writing exponential equations to represent the amount of interest earned over smaller intervals of time than annually, students will solidify their understanding of working with rational exponents in conjunction with the properties of exponents.

Core Standards Focus:

N.RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.

N.RN.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

c. Use the properties of exponents to transform expressions for exponential functions. For example the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.

F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

b. Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, and classify them as representing exponential growth or decay.

Related Standards: A.SSE.1

Launch (Whole Class):

Since this task is a continuation of the context and the mathematical work of the previous task, *Half Interested*, students should be able to begin work immediately on the task.

Explore (Small Group):

Students should draw upon their understanding of positive whole number exponents to make sense of the work of this task. Listen for how they reason about each of the problems, and if they can relate each to the following properties of exponents:

1. $a^m \cdot a^n = a^{m+n}$
2. $\left(a^m\right)^n = a^{mn}$
3. $(ab)^n = a^n \cdot b^n$
4. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
5. $\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$
6. $a^{-n} = \frac{1}{a^n}$

Identify any of these rules that seem problematic for students to discuss during the whole class discussion.

Discuss (Whole Class):

If necessary, illustrate each of these properties of exponents with examples using positive integer exponents, such as $2^3 \cdot 2^5 = (2\cdot 2\cdot 2)(2\cdot 2\cdot 2\cdot 2\cdot 2) = 2^8$ or $\frac{2^5}{2^3} = \frac{2\cdot 2\cdot 2\cdot 2\cdot 2}{2\cdot 2\cdot 2} = 2^2$. The question now is, "Do these properties that were developed with positive integer exponents still hold when the exponents are rational numbers?" The answer to this question is "yes" if rational number exponents are defined to mean $a^{m/n} = \left(\sqrt[n]{a}\right)^m$.

Here are possible responses for questions 11-13:

11. $A = 300(1.05)^{t/12}$ or $A = 300(1.004)^t$ or $A = 300 \cdot \sqrt[12]{1.05}^{\,t}$

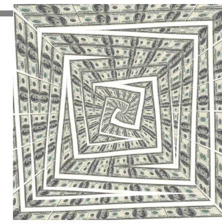
12. $A = 300(1.05)^{t/365}$ or $A = 300(1.0001337)^t$ or $A = 300 \cdot \sqrt[365]{1.05}^{\,t}$

13. $A = 382.88(1.05)^{-t/12}$ or $A = 382.88(.996)^t$

Point out that the base of exponential function is less than 1 in question 13. Examine the graph of this function as an example of exponential decay.

Aligned Ready, Set, Go: Solving Quadratic and Other Equations 3.3

Name: Solving Quadratic and Other Equations 3.3



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Ready, Set, Go!

Ready

Topic: Meaning of Exponents

In the table below there is a column for the exponential form, the meaning of that form, which is a list of factors and the standard form of the number. Fill in the form that is missing.

Exponential form	List of factors	Standard Form
5^3	$5 \cdot 5 \cdot 5$	125
1a.	$7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$	b.
2. 2^{10}	a.	b.
3a.	b.	81
4. 11^5	a.	b.
5a.	$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$	b.
6a.	b.	625

Provide at least three other equivalent forms of the exponential expression. Use rules of exponents such as $3^5 \cdot 3^6 = 3^{11}$ and $(5^2)^3 = 5^6$ as well as division properties and others.

	1st Equivalent Form	2nd Equivalent Form	3rd Equivalent Form
7. $2^{10} =$			
8. $3^7 =$			
9. $13^{-8} =$			
10. $7^{\frac{1}{3}} =$			
11. $5^1 =$			

Set

Topic: Finding equivalent expressions and functions.

Determine whether all three expressions in each problem below are equivalent. Justify why or why they are not equivalent.

12.	$5(3^{x-1})$	$15(3^{x-2})$	$\frac{3}{5}(3^{x})$
13.	$64\,(2^{-x})$	$\frac{64}{2^{x}}$	$64\left(\frac{1}{2}\right)^{x}$
14.	3(x-1)+4	3x - 1	3(x-2) +7
15.	$50(2^{x+2})$	$25(2^{2x+1})$	$50(4^{x})$
16.	$30(1.05^{x})$	$30\left(1.05^{\frac{1}{7}}\right)^{7x}$	$30\left(1.05^{\frac{x}{2}}\right)^{2}$
17.	$20\,(1.1^{x})$	$20\,(1.1^{-1})^{-1x}$	$20\left(1.1^{\frac{1}{5}}\right)^{5x}$

Go

Topic: Using rules of exponents

Simplify each expression. Your answer should still be in exponential form.

18. $7^{3}\cdot 7^{5}\cdot 7^{2}$

19. $(3^{4})^{5}$

20. $(5^{3})^{4}\cdot 5^{7}$

21. $x^{3}\cdot x^{5}$

22. x^{-b}

23. $x^{a}\cdot x^{b}$

24. $(x^{a})^{b}$

25. $\frac{y^{a}}{y^{b}}$

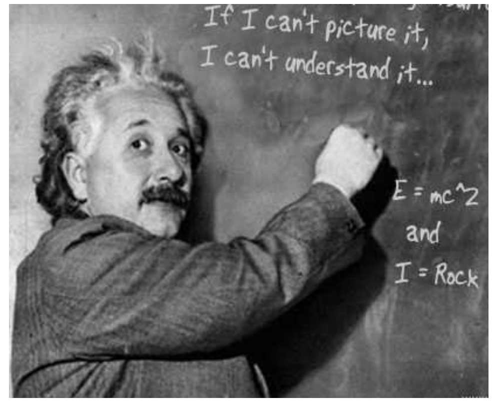
26. $\frac{(y^{a})^{c}}{y^{b}}$

27. $\frac{(3^{4})^{6}}{3^{7}}$

28. $\frac{r^{5}s^{3}}{rs^{2}}$

29. $\frac{x^{5}y^{12}z^{0}}{x^{8}y^{9}}$

3.4 Radical Ideas
A Practice Understanding Task



Now that Tia and Tehani know that $a^{m/n} = \left(\sqrt[n]{a}\right)^m$ they are wondering which form, radical form or exponential form, is best to use when working with numerical and algebraic expressions.

Tia says she prefers radicals since she understands the following properties for radicals (and there are not too many properties to remember):

If *n* is a positive integer greater than 1 and both *a* and *b* are positive real numbers then,

1. $\sqrt[n]{a^n} = a$
2. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
3. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Tehania says she prefers exponents since she understands the following properties for exponents (and there are more properities to work with):

1. $a^m \cdot a^n = a^{m+n}$
2. $\left(a^m\right)^n = a^{mn}$
3. $(ab)^n = a^n \cdot b^n$
4. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
5. $\dfrac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$
6. $a^{-n} = \dfrac{1}{a^n}$

DO THIS: Illustrate with examples and explain, using the properties of radicals and exponents, why $a^{1/n} = \sqrt[n]{a}$ and $a^{m/n} = \left(\sqrt[n]{a}\right)^m$ are true identities.

Using their preferred notation, Tia might simplify $\sqrt[3]{x^8}$ as follows:

$$\sqrt[3]{x^8} = \sqrt[3]{x^3 \cdot x^3 \cdot x^2} = \sqrt[3]{x^3} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{x^2} = x \cdot x \cdot \sqrt[3]{x^2} = x^2 \cdot \sqrt[3]{x^2}$$

(Tehani points out that Tia also used some exponent rules in her work.)

On the other hand, Tehani might simplify $\sqrt[3]{x^8}$ as follows:

$$\sqrt[3]{x^8} = x^{8/3} = x^{2+2/3} = x^2 \cdot x^{2/3} \text{ or } x^2 \cdot \sqrt[3]{x^2}$$

For each of the following problems, simplify the expression in the ways you think Tia and Tehani might do it.

Original expression	**What Tia and Tehani might do to simplify the expression:**
$\sqrt{27}$	Tia's method
	Tehani's method
$\sqrt[3]{32}$	Tia's method
	Tehani's method
$\sqrt{20x^7}$	Tia's method
	Tehani's method
$\sqrt[3]{\dfrac{16xy^5}{x^7y^2}}$	Tia's method
	Tehani's method

Tia and Tehani continue to use their preferred notation when solving equations.

For example, Tia might solve the equation $(x+4)^3 = 27$ as follows:

$$(x+4)^3 = 27$$
$$\sqrt[3]{(x+4)^3} = \sqrt[3]{27} = \sqrt[3]{3^3}$$
$$x+4 = 3$$
$$x = -1$$

Tehani might solve the same equation as follows:

$$(x+4)^3 = 27$$
$$\left[(x+4)^3\right]^{1/3} = 27^{1/3} = (3^3)^{1/3}$$
$$x+4 = 3$$
$$x = -1$$

For each of the following problems, simplify the expression in the ways you think Tia and Tehani might do it.

Original equation	What Tia and Tehani might do to solve the equation:
$(x-2)^2 = 50$	Tia's method
	Tehani's method
$9(x-3)^2 = 4$	Tia's method
	Tehani's method

Zac is showing off his new graphing calculator to Tia and Tehani. He is particularly excited about how his calculator will help him visualize the solutions to equations.

"Look," Zac says. "I treat the equation like a system of two equations. I set the expression on the left equal to y_1 and the expression of the right equal to y_2, and I know at the x value where the graphs intersect the expressions are equal to each other."

Zac shows off his new method on both of the equations Tia and Tehani solved using the properties of radicals and exponents. To everyone's surprise, both equations have a second solution.

1. Use Zac's graphical method to show that both of these equations have two solutions. Determine the exact values of the solutions you find on the calculator that Tia and Tehani did not find using their algebraic methods.

Tia and Tehani are surprised when they realize that both of these equations have more than one answer.

2. Explain why there is a second solution to each of these problems.

3. Modify Tia's and Tehani's algebraic approaches so they will find both solutions.

4. Use Zac's graphing calculator approach to solve the following problem.

   Carlos and Clarita deposited $300 in an account earning 5% interest. They want to take the money out of the account when it has doubled in value. To the nearest month, when can they withdraw their money?

3.4 Radical Ideas – Teacher Notes
A Practice Understanding Task

Purpose: This task provides opportunities for students to become fluent converting between exponential and radical representations of expressions, as well as using the rules of exponents to simplify exponential and radical expressions and to solve equations containing exponents.

Core Standards Focus:

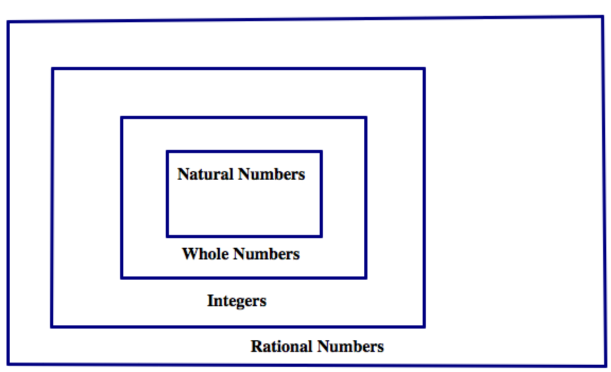
N.RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.

N.RN.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Mathematics II note for N.RN.1, N.RN.2
Exponents are extended from the integer exponents found in Mathematics I to rational exponents focusing on those that represent square or cube roots.

Related Standards: A.SSE.1

Launch (Whole Class):
Draw a Venn diagram of the relationship of subsets of the rational numbers as follows:



With student input, list a few examples of numbers that fit within each set. Point out to students that their understanding of which of these numbers can meaningfully be used as exponents has expanded over their years of experience working with exponents. Initially, only natural numbers made sense as exponents, since the exponent represented how many times the base was used as a factor (e.g., $5^3 = 5 \times 5 \times 5$).

Later, we expanded the set of reasonable exponents to include 0 and the negative integers. The reasonableness of treating such numbers as exponents was reinvestigated in the task *Half Interested* where negative integer exponents were used to represent an interest-bearing account balance *n years ago* rather than *n years from now*, giving meaning to 1.05^{-3} as dividing by the factor 1.05 three times (or multiplying by the factor $\frac{1}{1.05}$ three times). In that context, 0 would be used as an exponent to represent the current amount in the account when no time has elapsed. Since we want the account balance to stay the same at time $t = 0$, the factor $(1.05)^0$ would have to equal 1. Point out that this interpretation of 0 as an exponent is also consistent with the properties of exponents observed when using only natural numbers as exponents. For example, we want the additive property of exponents, $a^m \cdot a^n = a^{m+n}$, which was evident as a true observation for natural number exponents, to still hold with the extended set of exponents. Consequently, $a^n \cdot a^{-n} = a^0$ and $a^n \cdot a^{-n} = \frac{a^n}{a^n} = 1$, which implies $a^0 = 1$ in order to maintain consistency with the properties of exponents.

Now we have extended the set of reasonable exponents to include rational numbers. As we have observed in the past few tasks, the properties of exponents remain consistent if we define $a^{1/n} = \sqrt[n]{a}$ For example, $a^{1/3} \cdot a^{1/3} \cdot a^{1/3} = a^{1/3+1/3+1/3} = a^1 = a$ based on the properties of exponents in the same way as $\sqrt[3]{a} \cdot \sqrt[3]{a} \cdot \sqrt[3]{a} = a$ based on the definition of radicals.

[Note: Similar to the way we have extended the set of reasonable exponents to include integers and rational numbers when the need arose in terms of working with exponential equations, we will also need to extend the set of numbers we use when solving quadratic equations in future tasks. Introducing this idea of purposefully extending the numbers we give meaning to in a specific context will help set the stage for the expansion of the number system to include complex numbers in the next learning cycle of tasks. The Venn diagram used in this task will be extended to include the rest of the real numbers—the irrational numbers—and then the number system will be extended farther to include the real numbers as a subset of the complex numbers.]

After this brief review of how we have extended the meaning of exponents to include integer and rational exponents, review all of the properties of radicals and exponents listed on the first page of the task, and give students a few minutes to work on the DO THIS exploration outlined below the properties. Allow students to share their examples showing connections between the properties of radicals and exponents. After sharing a few examples, discuss Tia' and Tehani's preferred methods and have students begin working on the problems in the task.

Explore (Small Group):
If necessary, suggest that students might want to decompose numbers under the radical into their prime factorizations, or perhaps look for ways to decompose a number into factors that include perfect squares or perfect cubes, as needed. Listen for students who can make connections between Tia' and Tehani's strategies, such as Tia decomposes factors into powers of *n* (e.g., perfect

squares or cubes), while Tehani divides exponents to find how many groups of factors of size *n* can be formed, and how many factors are left over as factors in the radicand.

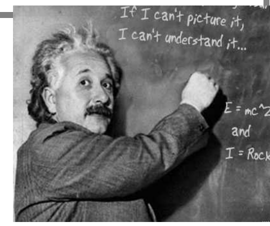
As you monitor student work, watch for any algebraic procedures that need to be discussed as a whole class.

Discuss (Whole Class):
As needed, discuss the algebra of specific problems. Most questions can be resolved by rewriting expression involving radicals as expressions involving rational exponents and using the properties of exponents to simplify the expression. Help students see the power of exponential form. Other techniques for working with radicals might surface and can be discussed; for example, looking for ways to meaningfully decompose the expression in the radicand.

Be sure to discuss Zac's graphical method since this reappears as a strategy for solving equations in future tasks.

Aligned Ready, Set, Go: Solving Quadratic and Other Equations 3.4

Name: Solving Quadratic and Other Equations 3.4



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Ready, Set, Go!

Ready

Topic: Standard form or Quadratic form

In each of the quadratic equations, $ax^2 + bx + c = 0$ **identify the values of a, b and c .**

1. $x^2 + 3x + 2 = 0$

   a =

   b =

   c =

2. $2x^2 + 3x + 1 = 0$

   a =

   b =

   c =

3. $x^2 - 4x - 12 = 0$

   a =

   b =

   c =

Write each of the quadratic expressions in factored form.

4. $x^2 + 3x + 2$

5. $2x^2 + 3x + 1$

6. $x^2 - 4x - 12$

7. $x^2 - 3x + 2$

8. $x^2 - 5x - 6$

9. $x^2 - 4x + 4$

10. $x^2 + 8x - 20$

11. $x^2 + x - 12$

12. $x^2 - 7x + 12$

Set

Topic: Radical notation and rational exponents

Each of the expressions below can be written using either radical notation, $\sqrt[n]{a^m}$ or rational exponents $a^{\frac{m}{n}}$. Rewrite each of the given expressions in the form that is missing. Express in most simplified form.

	Radical Form	Exponential Form
13.	$\sqrt[3]{5^2}$	
14.		$16^{\frac{3}{4}}$
15.	$\sqrt[3]{5^7 \cdot 3^5}$	
16.		$9^{\frac{2}{3}} \cdot 9^{\frac{4}{3}}$
17.	$\sqrt[5]{x^{13}y^{21}}$	
18.	$\sqrt[3]{27a^5b^2}$	
19.	$\sqrt[5]{\dfrac{32x^{13}}{243y^{15}}}$	
20.		$9^{\frac{3}{2}}s^{\frac{6}{3}}t^{\frac{1}{2}}$

Solve the equations below, use radicals or rational exponents as needed.

21. $(x+5)^4 = 81$

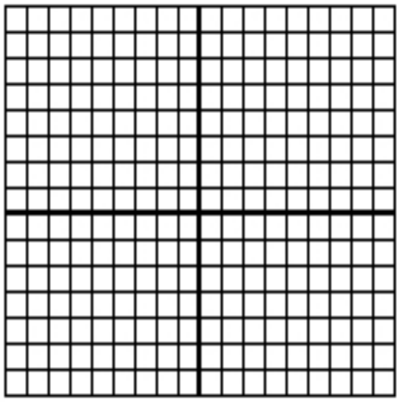
22. $2(x-7)^5 + 3 = 67$

Go

Topic: x-intercepts and y-intercepts for linear, exponential and quadratic

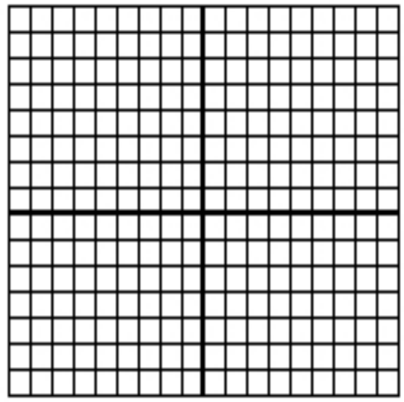
Given the function, find the x-intercept (s) and y-intercept if they exist and then use them to graph a sketch of the function.

23. $f(x) = (x+5)(x-4)$



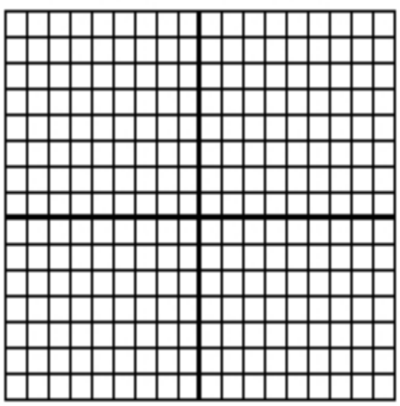
a. x-intercept(s): b. y-intercept:

24. $g(x) = 5(2^{x-1})$



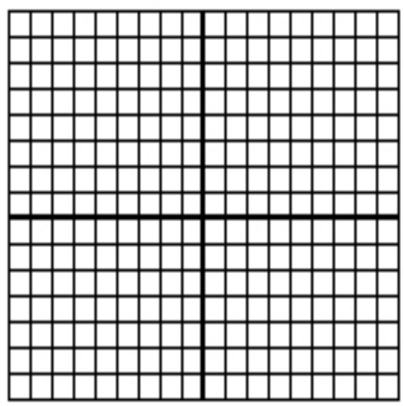
a. x-intercept(s): b. y-intercept:

25. $h(x) = -2(x+3)$



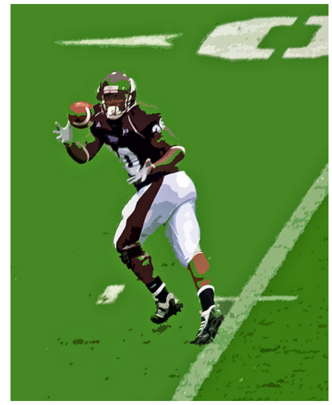
a. x-intercept(s): b. y-intercept:

26. $k(x) = x^2 - 4$



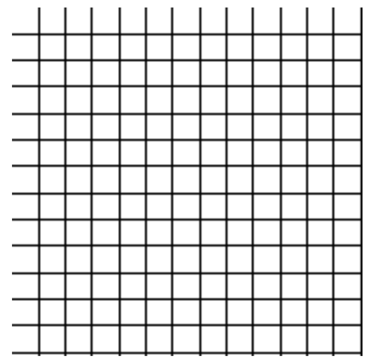
a. x-intercept(s): b. y-intercept:

3.5 Throwing an Interception
A Develop Understanding Task



The *x*-intercept(s) of the graph of a function $f(x)$ are often very important because they are the solution to the equation $f(x)=0$. In past tasks, we learned how to find the *x*-intercepts of the function by factoring, which works great for some functions, but not for others. In this task we are going to work on a process to find the *x*-intercepts of any quadratic function that has them. We'll start by thinking about what we already know about a few specific quadratic functions and then use what we know to generalize to all quadratic functions with *x*-intercepts.

1. What can you say about the graph of the function $f(x)=x^2-2x-3$?
   a. Graph the function



   b. What is the equation of the line of symmetry?

   c. What is the vertex of the function?

2. Now let's think specifically about the *x*-intercepts.
   a. What are the *x*-intercepts of $f(x)=x^2-2x-3$?

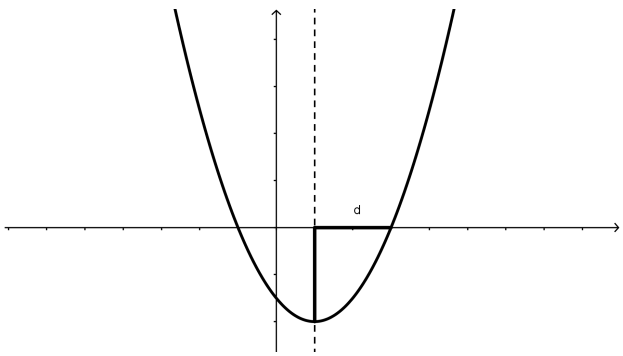
   b. How far are the *x*-intercepts from the line of symmetry?

   c. If you knew the line of symmetry was the line $x = h$, and you know how far the *x*-intercepts are from the line of symmetry, how would you find the actual *x*-intercepts?

   d. How far above the vertex are the *x*-intercepts?

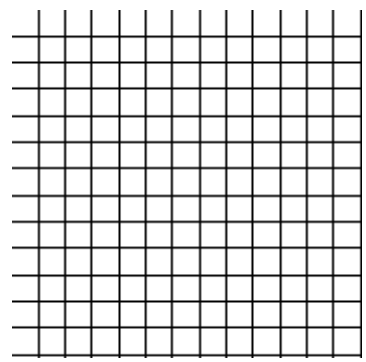
   e. What is the value of $f(x)$ at the *x*-intercepts?

Just to make it a little easier to talk about some of the features that relate to the intercepts, let's name them with variables. From now on, when we talk about the distance from the line of symmetry to either of the x intercepts, we'll call it *d*. The diagram below shows this feature.



We will always refer to the line of symmetry as the line $x = h$, so the two x-intercepts will be at the points $(h - d, 0)$ and $(h + d, 0)$.

3. So, let's think about another function: $f(x) = x^2 - 6x + 4$
   a. Graph the function by putting the equation into vertex form.



   b. What is the vertex of the function?

   c. What is the equation of the line of symmetry?

   d. What do you estimate the x-intercepts of the function to be?

   e. What do you estimate d to be?

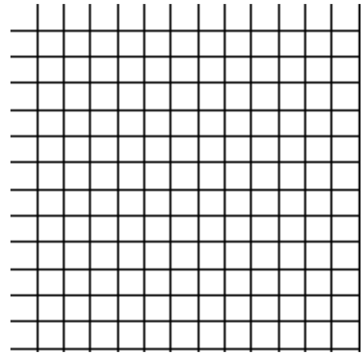
   f. What is the value of $f(x)$ at the x-intercepts?

   g. Using the vertex form of the equation and your answer to "f" above, write an equation and solve it to find the exact values of the x intercepts.

   h. What is the exact value of d?

   i. Use a calculator to find approximations for the x-intercepts. How do they compare with your estimates?

4. What about a function with a vertical stretch? Can we find exact values for the x-intercepts the same way? Let's try it with: $f(x) = 2x^2 - 8x + 5$.



a. Graph the function by putting the equation into vertex form.

b. What is the vertex of the function?

c. What is the equation of the line of symmetry?

d. What do you estimate the x-intercepts of the function to be?

e. What do you estimate d to be?

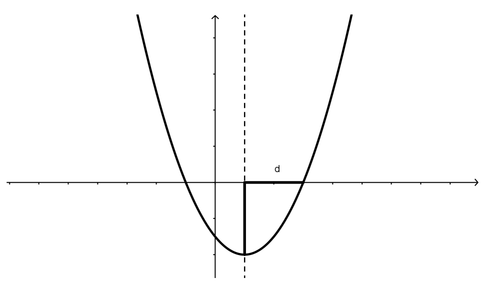
f. What is the value of $f(x)$ at the x-intercepts?

g. Using the vertex form of the equation and your answer to "f" above, write an equation and solve it to find the exact values of the x-intercepts.

h. What is the exact value of d?

i. Compare your solution to your estimate of the roots. How did you do?

5. Finally, let's try to generalize this process by using: $f(x) = ax^2 + bx + c$ to represent any quadratic function that has x-intercepts. Here's a possible graph of $f(x)$.



a. Start the process the usual way by putting the equation into vertex form. It's a little tricky, but just do the same thing with a, b, and c as what you did in the last problem with the numbers.

b. What is the vertex of the parabola?

c. What is the line of symmetry of the parabola?

d. Write and solve the equation for the x-intercepts just as you did previously.

6. How could you use the solutions you just found to tell what the x-intercepts are for the function $f(x) = x^2 - 3x - 1$?

7. You may have found the algebra for writing the general quadratic function $f(x) = ax^2 + bx + c$ in vertex form a bit difficult. Here is another way we can work with the general quadratic function leading to the same results you should have arrived at in 5d.

   a. Since the two x-intercepts are d units from the line of symmetry $x = h$, if the quadratic crosses the x-axis its x-intercepts are at $(h - d, 0)$ and $(h + d, 0)$. We can always write the factored form of a quadratic if we know its x-intercepts. The factored form will look like $f(x) = a(x - p)(x - q)$ where p and q are the two x-intercepts. So, using this information, write the factored form of the general quadratic $f(x) = ax^2 + bx + c$ using the fact that its x-intercepts are at h-d and h+d.

   b. Multiply out the factored form (you will be multiplying two **trinomial** expressions together) to get the quadratic in standard form. Simplify your result as much as possible by combining like terms.

c. You now have the same general quadratic function written in standard form in two different ways, one where the **coefficients** of the terms are *a*, *b* and *c* and one where the coefficients of the terms are expressions involving *a*, *h* and *d.* Match up the coefficients; that is, *b,* the coefficient of *x* in one version of the standard form is equivalent to ________ in the other version of the standard form. Likewise *c*, the constant term in one version of the standard form is equivalent to _________ in the other.

d. Solve the equations *b* = ________ and *c* = ________ for *h* and *d*. Work with your equations until you can express *h* and *d* with expressions that only involve *a, b* and *c.*

e. Based on this work, how can you find the *x*-intercepts of any quadratic using only the values for *a, b* and *c*?

f. How does your answer to "e" compare to your result in 5d?

8. All of the functions that we have worked with on this task have had graphs that open upward. Would the formula work for parabolas that open downward? Tell why or why not using an example that you create using your own values for the coefficients *a*, *b*, and *c.*

3.5 Throwing an Interception – Teacher Notes

A Develop Understanding Task

Purpose: The purpose of this task is to develop the quadratic formula as a way of finding *x*-intercepts of a quadratic function that crosses the *x*-axis. In a future task this same quadratic formula will be used to find the roots of any quadratic, including those with complex roots whose graphs do not cross the *x*-axis. In this task, the quadratic formula is developed from the perspective of visualizing the distance the *x*-intercepts are away from the axis of symmetry. Therefore, if the axis of symmetry is the line $x = h$, and the *x*-intercepts are d units from the axis of symmetry, then the coordinates of the *x*-intercepts are $(h - d, 0)$ and $(h + d, 0)$. This fact is used to develop the quadratic formula from one perspective (see problem 7). The quadratic formula is also developed from the perspective of writing the general quadratic $f(x) = ax^2 + bx + c$ in vertex form by completing the square (see problem 5). The quadratic formula students develop in this task will probably look like $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$, written with two terms, one term to represent the location of the axis of symmetry, and the other term to represent the distance the *x*-intercepts are away from the axis of symmetry. You might want to discuss how this can be more generally written as a single term, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Core Standards Focus:

A.REI.4 Solve quadratic equations in one variable.

- Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

A.CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

Related Standards:

Launch (Whole Class):

In this task, students will draw heavily upon their work from module 2, particularly changing the form of a quadratic from standard form to vertex form. You may want to review that procedure with students before beginning this task, particularly reviewing an example of completing the square when the coefficient of the x^2 term is not 1. Here are some additional algebraic ideas that are used in this task that would be good to go over before students encounter these ideas in the task:

- We know how to multiply two binomial expressions to get a trinomial expression. How might we extend this process to multiplying two trinomial expressions?
- What algebra is implied by a squared-binomial term?
- How do we algebraically move between a perfect-square-trinomial and a squared-binomial?
- How do we complete the square when a trinomial is not a perfect-square?

Make sure that students are familiar with the vocabulary terms *binomial*, *trinomial* and *coefficient.* Point out the note at the beginning of this task: that the work we are doing in this task to find the *x*-intercepts of a quadratic function will help us solve quadratic equations where the quadratic expression is set equal to 0.

Explore (Small Group):
Listen for students who are surfacing the idea that the *x*-intercepts of a quadratic function are equidistant from the axis of symmetry. Watch for how they are using this idea in their work on questions 1-4.

Question 5 may prove somewhat difficult for students as they work on writing the general quadratic function $f(x) = ax^2 + bx + c$ in vertex form $f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$. You may want to work through this together as a class and then let students return to the work of letting $f(x) = 0$ and solving for x. Identify students who can discuss the algebra of question 5 for the whole class discussion.

Question 6 is intended to give students an opportunity to apply the quadratic formula obtained in question 5 to a specific case. Identify students who can discuss this problem in the whole class discussion.

Question 7 provides an alternative algebraic approach for deriving the quadratic formula that does not include completing the square. Instead, students make use of the idea that the *x*-intercepts are *d* units from the axis of symmetry *x* = *h* and therefore, are located at *h* – *d* and *h* + *d.* Using the *x*-intercepts we can write the factored form of the function as $f(x) = a(x - h + d)(x - h - d)$. Multiplying out these trinomials leads to $f(x) = ax^2 - 2ahx + ah^2 - ad^2$. Matching the coefficients of the terms of this expression to the *a, b* and *c* of the standard form yields two equations: $b = -2ah$ and $c = ah^2 - ad^2$. Solving the first equation for *h* results in $h = \frac{-b}{2a}$. Substituting this expression for *h* in the second equation and then solving for *d* yields $d = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$. Since the *x*-intercepts are *d* units from *x* = *h* they must be located at $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$.

Discuss (Whole Class):
The whole class discussion should focus on the key idea of the *x*-intercepts being *d* units from the axis of symmetry $x = h$ and how this leads to a formula for finding the *x*-intercepts of any quadratic using only the coefficients *a, b* and *c.*

If needed, have a student present question 4 to illustrate the process of finding the *x*-intercepts using this process for a specific case. Then have a student present their work on question 5 for the general case. Have another student present their work on question 7 to illustrate an alternative algebraic method for finding the quadratic form (see the outline of this work in the explore notes).

Make sure to discuss question 6 so all students can use the quadratic formula regardless of whether they were successful in deriving it in question 5 or question 7.

If there is time, discuss some of the student examples for question 8. However, it is not necessary to discuss this question.

Aligned Ready, Set, Go: Solving Quadratic and Other Equations 3.5

Name: Solving Quadratic and Other Equations | 3.5

Ready, Set, Go!

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Ready

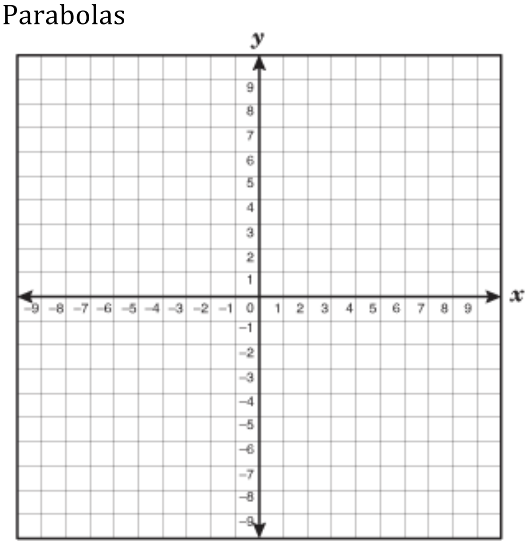
Topic: Converting measurement of area, area and perimeter.

While working with areas it sometimes essential to convert between units of measure, for example changing from square yards to square feet and so forth. Convert the areas below to the desired measure. (Hint: area is two dimensional, for example 1 yd^2 = 9 ft^2 because 3 ft along each side of a square yard equals 9 square feet.)

1. 7 yd^2 = ? ft^2
2. 5 ft^2 = ? in^2
3. 1 $mile^2$ = ? ft^2
4. 100 m^2 = ? cm^2
5. 300 ft^2 = ? yd^2
6. 96 in^2 = ? ft^2

Set

Topic: Transformations and Parabolas, Symmetry and Parabolas



7a. Graph each of the quadratic functions.

$$f(x) = x^2$$
$$g(x) = x^2 - 9$$
$$h(x) = (x+2)^2 - 9$$

b. How do the functions compare to each other?

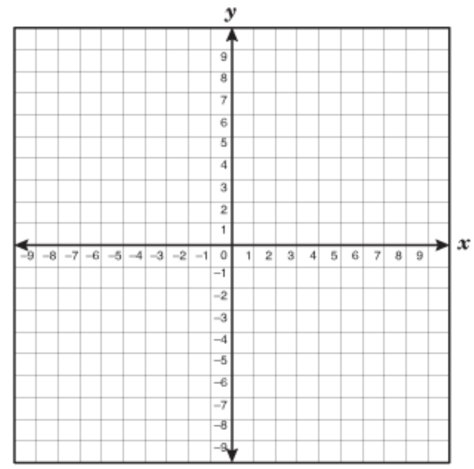
c. How do *g(x)* and *h(x)* compare to *f(x)*?

d. Look back at the functions above and identify the x-intercepts of *g(x).* What are they?

e. What are the coordinates of the points corresponding to the x-intercepts in *g(x)* in each of the other functions*?* How do these coordinates compare to one another?

8a. Graph each of the quadratic functions.

$$f(x) = x^2$$
$$g(x) = x^2 - 4$$
$$h(x) = (x-1)^2 - 4$$



b. How do the functions compare to each other?

c. How do *g(x)* and *h(x)* compare to *f(x)*?

d. Look back at the functions above and identify the x-intercepts of *g(x).* What are they?

e. What are the coordinates of the points corresponding to the x-intercepts in *g(x)* in each of the other functions*?* How do these coordinates compare to one another?

9. How can the transformations that occur to the function $f(x) = x^2$ be used to determine where the x-intercepts of the function's image will be?

Go

Topic: Function Notation and Evaluating Functions

Use the given functions to find the missing values. (Check your work using a graph.)

10. $f(x) = x^2 + 4x - 12$

a. $f(0) =$ ______

b. $f(2) =$ _____

c. $f(x) = 0, \quad x =$ _____

d. $f(x) = 20, \quad x =$ _____

11. $g(x) = (x-5)^2 + 2$

a. $g(0) =$ _____

b. $g(5) =$ ______

c. $g(x) = 0, \quad x =$ ______

d. $g(x) = 16, \quad x =$ ______

12. $f(x) = x^2 - 6x + 9$

a. $f(0) =$ ______

b. $f(-3) =$ ______

c. $f(x) = 0$, $x =$ ______

d. $f(x) = 16$, $x =$ ______

13. $g(x) = (x-2)^2 - 3$

a. $g(0) =$ ______

b. $g(5) =$ ______

c. $g(x) = 0$, $x =$ ______

d. $g(x) = -3$, $x =$ ______

14. $f(x) = (x+5)^2$

a. $f(0) =$ ______

b. $f(-2) =$ ______

c. $f(x) = 0$, $x =$ ______

d. $f(x) = 9$, $x =$ ______

15. $g(x) = -(x+1)^2 + 8$

a. $g(0) =$ ______

b. $g(2) =$ ______

c. $g(x) = 0$, $x =$ ______

d. $g(x) = 4$, $x =$ ______

3.6 Curbside Rivalry

A Solidify Understanding Task

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Carlos and Clarita have a brilliant idea for how they will earn money this summer. Since the community in which they live includes many high schools, a couple of universities, and even some professional sports teams, it seems that everyone has a favorite team they like to root for. In Carlos' and Clarita's neighborhood these rivalries take on special meaning, since many of the neighbors support different teams. They have observed that their neighbors often display handmade posters and other items to make their support of their favorite team known. The twins believe they can get people in the neighborhood to buy into their new project: painting team logos on curbs or driveways.

For a small fee, Carlos and Clarita will paint the logo of a team on a neighbor's curb, next to their house number. For a larger fee, the twins will paint a mascot on the driveway. Carlos and Clarita have designed stencils to make the painting easier and they have priced the cost of supplies. They have also surveyed neighbors to get a sense of how many people in the community might be interested in purchasing their service. Here is what they have decided, based on their research.

- *A curbside logo will require 48 in^2 of paint*
- *A driveway mascot will require 16 ft^2 of paint*
- *Surveys show the twins can sell 100 driveway mascots at a cost of $20, and they will sell 10 fewer mascots for each additional $5 they charge*

1. If a curbside logo is designed in the shape of a square, what will its dimensions be?

A square logo will not fit nicely on a curb, so Carlos and Clarita are experimenting with different types of rectangles. They are using a software application that allows them to stretch or shrink their logo designs to fit different rectangular dimensions.

2. Carlos likes the look of the logo when the rectangle in which it fits is 8 inches longer than it is wide. What would the dimensions of the curbside logo need to be to fit in this type of rectangle? As part of your work, write a quadratic equation that represents these requirements.

3. Clarita prefers the look of the logo when the rectangle in which it fits is 13 inches longer than it is wide. What would the dimensions of the curbside logo need to be to fit in this type of rectangle? As part of your work, write a quadratic equation that represents these requirements.

Your quadratic equations on the previous two problems probably started out looking like this: $x(x+n)=48$ where *n* represents the number of inches the rectangle is longer than it is wide. The expression on the left of the equation could be multiplied out to get and equation of the form $x^2+nx=48$. If we subtract 48 from both sides of this equation we get $x^2+nx-48=0$. In this form, the expression on the left looks more like the quadratic functions you have been working with in previous tasks, $y=x^2+nx-48$.

4. Consider Carlos' quadratic equation where *n* = 8, so $x^2+8x-48=0$. How can we use our work with quadratic functions like $y=x^2+8x-48$ to help us solve the quadratic equation $x^2+8x-48=0$? Describe at least two different strategies you might use, and then carry them out. Your strategies should give you solutions to the equation as well as a solution to the question Carlos is trying to answer in #2.

5. Now consider Clarita's quadratic equation where *n* = 13, so $x^2+13x-48=0$. Describe at least two different strategies you might use to solve this equation, and then carry them out. Your strategies should give you solutions to the equation as well as a solution to the question Clarita is trying to answer in #3.

6. After much disagreement, Carlos and Clarita agree to design the curbside logo to fit in a rectangle that is 6 inches longer than it is wide. What would the dimensions of the curbside logo need to be to fit in this type of rectangle? As part of your work, write and solve a quadratic equation that represents these requirements.

7. What are the dimensions of a driveway mascot if it is designed to fit in a rectangle that is 6 feet longer than it is wide? (See the requirements for a driveway mascot given in the bulleted list above.) As part of your work, write and solve a quadratic equation that represents these requirements.

8. What are the dimensions of a driveway mascot if it is designed to fit in a rectangle that is 10 feet longer than it is wide? (See the requirements for a driveway mascot given in the bulleted list above.) As part of your work, write and solve a quadratic equation that represents these requirements.

Carlos and Clarita are also examining the results of their neighborhood survey, trying to determine how much they should charge for a driveway mascot. Remember, this is what they have found from the survey: *They can sell 100 driveway mascots at a cost of $20, and they will sell 10 fewer mascots for each additional $5 they charge.*

9. Write an equation, make a table, and sketch a graph for the number of driveway mascots the twins can sell for each $5 increment, x, in the price of the mascot.

10. Write an equation, make a table, and sketch a graph (on the same set of axes) for the price of the driveway mascot for each $5 increment, x, in the price.

11. Write an equation, make a table, and sketch a graph for the revenue the twins will collect for each $5 increment in the price of the mascot.

12. The twins estimate that the cost of supplies will be $250 and they would like to make $2000 in profit from selling driveway mascots. Therefore, they will need to collect $2250 in revenue. Write and solve a quadratic equation that represents collecting $2250 in revenue. Be sure to clearly show your strategy for solving this quadratic equation.

3.6 Curbside Rivalry – Teacher Notes

A Solidify Understanding Task

Purpose: In this task students use their techniques for changing the forms of quadratic expressions (i.e., factoring, completing the square to put the quadratic in vertex form, or using the quadratic formula to find the x-intercepts) as strategies for solving quadratic equations.

Core Standards Focus:

A.REI.4 Solve quadratic equations in one variable.

a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x – p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as a ± bi for real numbers a and b.

Note for Mathematics II A.REI.4a, A.REI.4b

Extend to solving any quadratic equation with real coefficients, including those with complex solutions.

A.REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line $y = –3x$ and the circle $x^2 + y^2 = 3$.

Note for Mathematics II A.REI.7

Include systems consisting of one linear and one quadratic equation. Include systems that lead to work with fractions.

A.CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. For example, rearrange Ohm's law V = IR to highlight resistance R.

Related Standards: A.SSE.1

Launch (Whole Class):

Before starting this task, it will be helpful to point out some terminology used with quadratics. In previous tasks we have been working with quadratic functions, $f(x) = ax^2 + bx + c$. In this task we will be working with quadratic equations, $ax^2 + bx + c = 0$. We will also refer to quadratic expressions, $ax^2 + bx + c$. Introduce students to the words *roots* and *zeroes* as ways of referring to the *x*-values that are solutions to a quadratic equation or the *x*-values that make a quadratic expression zero. These words can also be used to refer to the *x*-intercepts of a quadratic function that crosses the *x*-axis.

Point out to students that the goal of this task is to learn how to write and solve quadratic equations that arise from different problem situations, and that they will experiment with ways of using the form-changing techniques of previous tasks to support the work of solving quadratic equations.

As part of the launch, read through the context of the task and have students work on question 1 where they will write a simple quadratic equation, $x^2 = 48$ to represent the context. Make sure that students understand they can solve for *x* by taking the square root of both sides of this equation. They developed strategies for working with such radical expressions in task 3.4 *Radical Ideas*. Point out that while there are two numbers we can square to get 48, only the positive square root of 48 makes sense in this context. Also, for the purpose of the context, decimal approximations for square roots provide reasonable solutions.

Also start problem 2 together, before setting students to work on the task. Help students recognize that a quadratic equation that would represent this situation would be $x(x+8) = 48$. Ask students how they might solve such an equation. One method they might suggest would be guess and check. Another method might be to graph the quadratic $y = x(x+8)$ and the line $y = 48$ and look for their points of intersection as Zac did in task 3.4. Point out that the task will help them think about more strategies, particularly algebraic strategies, that they might use on these types of problems.

Explore (Small Group):
After problem 3 the task suggests a typical algebraic strategy that might be used to solve these types of quadratic equations. For example, to solve question 4, multiply out the quadratic expression on the left, and then subtract 48 from both sides to get $x^2 + 8x - 48 = 0$ as an equivalent equation. Solving this equation would be like trying to find the *x*-intercepts of the quadratic function $f(x) = x^2 + 8x - 48$. Ask students how they might find these *x*-intercepts. Try to press for two strategies: finding the factors and determining what values of *x* make the factors zero; or, using the quadratic formula from the previous task *Throwing an Interception.* Similar approaches will work for questions 5-8. Help students see that factoring is an effective strategy sometimes, but not all quadratic expressions factor nicely. The quadratic formula can always be used to find the solutions, but can be cumbersome to apply.

Questions 9-12 provide an opportunity to create and solve a quadratic equation that deals with optimization. Students write two linear equations to represent the number of mascots to be sold, $y = 100 - 10x$, and the price of each mascot, $y = 20 + 5x$. The product of these two functions, $y = (100 - 10x)(20 + 5x)$, represents the revenue collected. A typical question one might ask is to find the maximum revenue, which could be answered by finding the vertex of this function. In this task the question asked—when will the revenue equal $2250—leads to a quadratic equation to be solved: $2250=(100-10x)(20+5x)$. Again the strategy of changing the form of this equation to an equivalent quadratic equation where one side equals zero provides a path to a solution. Students may also recognize that the solution shows up in the table for revenue.

Discuss (Whole Class):

Focus the whole class discussion on this concept: Since the solutions to quadratic equations of the form $f(x)=0$ occur when the function crosses the *x*-axis, setting factors equal to 0 or using the quadratic formula are reasonable strategies for solving such equations. Select problems from the task that seem the most helpful for your students, including at least one problem that can be solved by factoring and one that requires the quadratic formula.

Given time, it would be good to discuss questions 9-12 to remind students that (1) quadratics are the product of two linear functions, (2) the *x*-intercepts of the quadratic function are the *x*-intercepts of the individual linear factors, and (3) the vertex of the quadratic is on the axis of symmetry halfway between the *x*-intercepts. It would be good to connect the graphical, numerical and algebraic ways the solutions to this problem get represented by examining the data in the table of the revenue, by graphing the revenue function and the horizontal line representing the desired revenue, and by solving this equation using the quadratic formula.

Aligned Ready, Set, Go: Solving Quadratic and Other Equations 3.6

Name: Solving Quadratic and Other Equations | 3.6

Ready, Set, Go!

Ready

Topic: Finding x-intercepts for linear equations.

1. Find the x-intercept of each equation below. Write your answer as an ordered pair. Consider how the format of the given equation either facilitates for inhibits your work.

a. $3x + 4y = 12$ b. $y = 5x - 3$ c. $y - 5 = -4(x + 1)$

d. $y = -4x + 1$ e. $y - 6 = 2(x + 7)$ f. $5x - 2y = 10$

2. Which of the linear equation formats above facilitates your work in finding x-intercepts? Why?

3. Using the same equations from question 1, find the y-intercepts. Write your answers as ordered pairs

a. $3x + 4y = 12$ b. $y = 5x - 3$ c. $y - 5 = -4(x + 1)$

d. $y = -4x + 1$ e. $y - 6 = 2(x + 7)$ f. $5x - 2y = 10$

4. Which of the formats above facilitate finding the y-intercept? Why?

Set

Topic: Solve Quadratic Equations, Connecting Quadratics with Area

For each of the given quadratic equations, (a) describe the rectangle the equation fits with. (b) What constraints have been placed on the dimensions of the rectangle?

5. $x^2 + 7x - 170 = 0$

6. $x^2 + 15x - 16 = 0$

7. $x^2 + 2x - 35 = 0$

8. $x^2 + 10x - 80 = 0$

Solve the quadratic equations below.

9. $x^2 + 7x - 170 = 0$

10. $x^2 + 15x - 16 = 0$

11. $x^2 + 2x - 35 = 0$

12. $x^2 + 10x - 80 = 0$

Go

Topic: Factoring Expressions

Write each of the expressions below in factored form.

13. $x^2 - x - 132$

14. $x^2 - 5x - 36$

15. $x^2 + 5x + 6$

16. $x^2 + 13x + 42$

17. $x^2 + x - 56$

18. $x^2 - x$

19. $x^2 - 8x + 12$

20. $x^2 - 10x + 25$

21. $x^2 + 5x$

Need Assistance? Check out these additional resources:

https://www.khanacademy.org/math/trigonometry/polynomial_and_rational/quad_factoring/v/factoring-quadratic-expressions

3.7 Perfecting My Quads

A Practice Understanding Task

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Carlos and Clarita, Tia and Tehani, and their best friend Zac are all discussing their favorite methods for solving quadratic equations of the form $ax^2 + bx + c = 0$. Each student thinks about the related quadratic function $y = ax^2 + bx + c$ as part of his or her strategy.

Carlos: "I like to make a table of values for *x* and find the solutions by inspecting the table."

Clarita: "I like to write the equation in factored form, and then use the factors to find the solutions."

Tia: "I like to treat it like a quadratic function that I am trying to put in vertex form by completing the square. I can then use a square root to undo the squared expression."

Tehani: "I also like to treat it like a quadratic function, but I use the quadratic formula to find the solutions."

Zac: "I like to graph the related quadratic function and use my graph to find the solutions."

Demonstrate how each student might solve each of the following quadratic equations.

Solve: $x^2 - 2x - 15 = 0$	Carlos' Strategy	Zac's Strategy
Clarita's Strategy	Tia's Strategy	Tehani's Strategy

Solve: $2x^2 + 5x - 12 = 0$	Carlos' Strategy	Zac's Strategy
Clarita's Strategy	Tia's Strategy	Tehani's Strategy

Solve: $x^2 + 4x - 8 = 0$	Carlos' Strategy	Zac's Strategy
Clarita's Strategy	Tia's Strategy	Tehani's Strategy

Solve: $8x^2 + 2x = 3$	Carlos' Strategy	Zac's Strategy
Clarita's Strategy	Tia's Strategy	Tehani's Strategy

Describe why each strategy works.

As the students continue to try out their strategies, they notice that sometimes one strategy works better than another. Explain how you would decide when to use each strategy.

Here is an extra challenge. How might each student solve the following system of equations?

Solve the system: $y_1 = x^2 - 4x + 1$ $y_2 = x - 3$	<u>Carlos' Strategy</u>	<u>Zac's Strategy</u>
<u>Clarita's Strategy</u>	<u>Tia's Strategy</u>	<u>Tehani's Strategy</u>

3.7 Perfecting My Quads – Teacher Notes
A Solidify Understanding Task

Purpose: In this task students use their techniques for changing the forms of quadratic expressions (i.e., factoring, completing the square to put the quadratic in vertex form, or using the quadratic formula to find the x-intercepts) as strategies for solving quadratic equations.

Core Standards Focus:

A.REI.4 Solve quadratic equations in one variable.

a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Note for Mathematics II A.REI.4a, A.REI.4b
Extend to solving any quadratic equation with real coefficients, including those with complex solutions.

A.REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.

Note for Mathematics II A.REI.7
Include systems consisting of one linear and one quadratic equation. Include systems that lead to work with fractions.

A.CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

Related Standards: A.SSE.1

Launch (Whole Class):
Remind students that in the previous task, *Curbside Rivalry,* they used various strategies to solve quadratic equations that arose from various problem situations that Carlos and Clarita were trying to resolve. In this task the focus is again on solving quadratic equations, but no contexts are provided. Instead, students are to try out several different strategies and procedures for solving the equations and to focus on the strengths and weaknesses of each method.

Read through the first part of the task handout with the class, and make sure they understand the basic strategy each of the characters in the story plan to use. Then set students to work trying out each of the strategies on a variety of problems.

Explore (Small Group):
As students work through the task they should notice that some strategies, such as factoring or making a table, do not work as consistently as some other strategies, although they are effective and easy to do when they do yield solutions. Encourage students to focus on the types of solutions that seem to support each method. For example, making a table works better when the solutions are integers, or at least rational numbers.

Discuss (Whole Class):
Focus the discussion on the questions "describe why each strategy works" and "explain how you would decide when to use each strategy."

Illustrate the value of the graphical and numerical strategies by working through the last problem, which involves a system of equations where one equation is quadratic and one equation is linear. Point out how the graph and table give us a sense of what a solution to this system would mean. Students may wonder about how to start an algebraic approach for these problems. Remind students that with systems of equations we can sometimes set the equations equal to each other. Doing so will lead to an equation that can be solved by rearranging the terms to get a quadratic expression equal to 0.

Aligned Ready, Set, Go: Solving Quadratic and Other Equations 3.7

Name: Solving Quadratic and Other Equations | 3.7

Ready, Set, Go!

Ready

Topic: Symmetry and Distance

The given functions provide the connection between possible areas, *A(x)* , that can be created by a rectangle for a given side length, *x*, and a set amount of perimeter. You could think of it as the different amounts of area you can close in with a given amount of fencing as long as you always create a rectangular enclosure.

1. $A(x) = x(10 - x)$

Find the following:

a. A(3) =   b. A(4) =

c. A(6) =   d. $A(x) = 0$

e. When is A(x) at its maximum? Explain or show how you know.

2. $A(x) = x(50 - x)$

Find the following:

a. A(10) =   b. A(20) =

c. A(30) =   d. A(x) = 0

e. When is A(x) at its maximum? Explain or show how you know.

3. $A(x) = x(75 - x)$

Find the following:

a. $A(20) =$ *b.* $A(35) =$

c. $A(40) =$ *d.* $A(x) = 0$

e. When is A(x) at its maximum? Explain or show how you know.

4. $A(x) = x(48 - x)$

Find the following:

a. A(10) =   b. A(20) =

c. A(28) =   d. A(x) = 0

e. When is A(x) at its maximum? Explain or show how you know.

Set

Topic: Solving Quadratic Equations Efficiently

For each of the given quadratic equations find the solutions using an efficient method. State the method you are using as well as the solutions. You must use at least three different methods.

5. $x^2 + 17x + 60 = 0$

6. $x^2 + 16x + 39 = 0$

7. $x^2 + 7x - 5 = 0$

8. $3x^2 + 14x - 5 = 0$

9. $x^2 - 12x = -8$

10. $x^2 + 6x = 7$

Summarize the process for solving a quadratic by the indicated strategy. Give examples along with written explanation, also indicate when it is best to use this strategy.

11. Completing the Square

12. Factoring

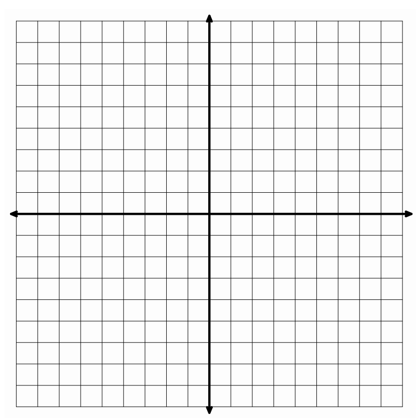
13. Quadratic Formula

Go

Topic: Graphing quadratics and finding essential features of the graph. Solving systems of equations.

Graph the quadratic function and supply the desired information about the graph.

14. $f(x) = x^2 + 8x + 13$



a. Line of symmetry:

b. x-intercepts:

c. y-intercept:

d. vertex:

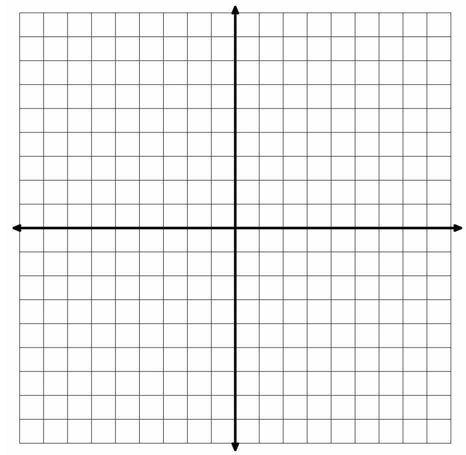
15. $f(x) = x^2 - 4x - 1$

a. Line of symmetry:

b. x-intercepts:

c. y-intercept:

d. vertex:



Solve each system of equations using an algebraic method and check your work!

16.
$$\begin{cases} 3x + 5y = 15 \\ 3x - 2y = 6 \end{cases}$$

17.
$$\begin{cases} y = -7x + 12 \\ y = 5x - 36 \end{cases}$$

18.
$$\begin{cases} y = 2x + 12 \\ y = 10x - x^2 \end{cases}$$

19.
$$\begin{cases} y = 24x - x^2 \\ y = 8x + 48 \end{cases}$$

3.8 To Be Determined . . .

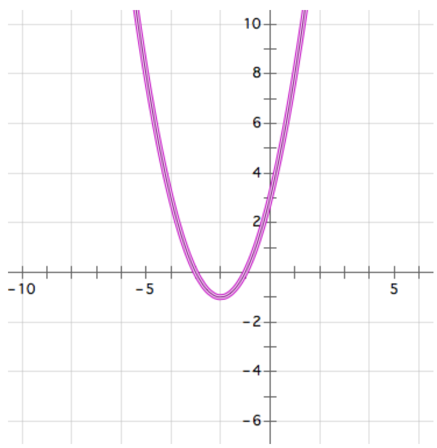
A Develop Understanding Task

Israel and Miriam are working together on a homework assignment. They need to write the equations of quadratic functions from the information given in a table or a graph. At first, this work seemed really easy. However, as they continued to work on the assignment, the algebra got more challenging and raised some interesting questions that they can't wait to ask their teacher.

Work through the following problems from Israel and Miriam's homework. Use the information in the table or the graph to write the equation of the quadratic function in all three forms. You may start with any form you choose, but you need to find all three equivalent forms. (If you get stuck, your teacher has some hints from Israel and Miriam that might help you.)

1.

x	y
-5	8
-4	3
-3	0
-2	-1
-1	0
0	3
1	8
2	15
3	24
4	35
5	48



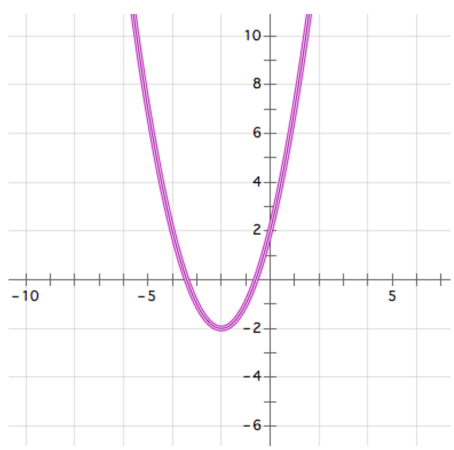
Standard form:

Factored form:

Vertex form:

2.

x	y
-5	7
-4	2
-3	-1
-2	-2
-1	-1
0	2
1	7
2	14
3	23
4	34
5	47



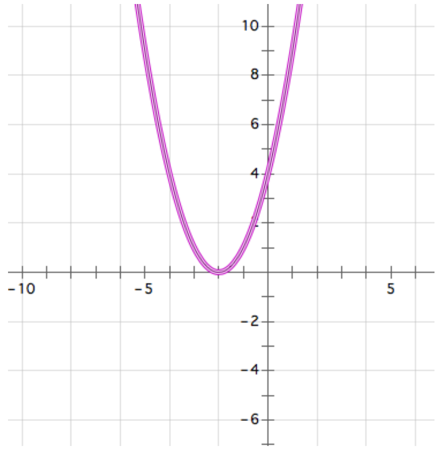
Standard form:

Factored form:

Vertex form:

3.

x	y
-5	9
-4	4
-3	1
-2	0
-1	1
0	4
1	9
2	16
3	25
4	36
5	49



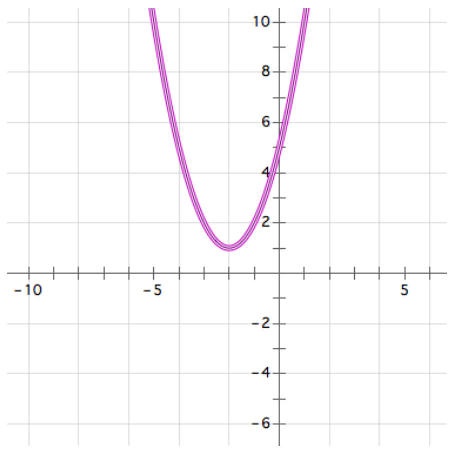
Standard form:

Factored form:

Vertex form:

4.

x	y
-5	10
-4	5
-3	2
-2	1
-1	2
0	5
1	10
2	17
3	26
4	37
5	50



Standard form:

Factored form:

Vertex form:

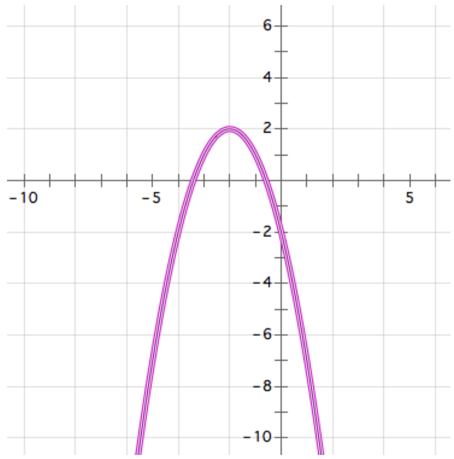
5. Israel was concerned that his factored form for the function in question 4 didn't look quite right. Miriam suggested that he test it out by plugging in some values for *x* to see if he gets the same points as those in the table. Test *your* factored form. Do you get the same values as those in the table?

6. Why might Israel be concerned about writing the factored form of the function in question 4?

Here are some more of Israel and Miriam's homework.

7.

x	y
-5	-7
-4	-2
-3	1
-2	2
-1	1
0	-2
1	-7
2	-14
3	-23
4	-34
5	-47



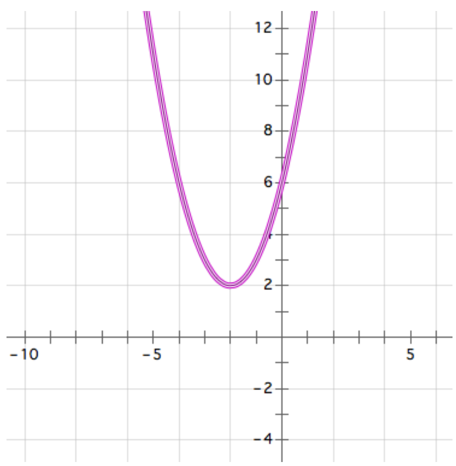
Standard form:

Factored form:

Vertex form:

8.

x	y
-5	11
-4	6
-3	3
-2	2
-1	3
0	6
1	11
2	18
3	27
4	38
5	51



Standard form:

Factored form:

Vertex form:

9. Miriam notices that the graphs of function 7 and function 8 have the same vertex point. Israel notices that the graphs of function 2 and function 7 are mirror images across the *x*-axis. What do you notice about the roots of these three quadratic functions?

The Fundamental Theorem of Algebra

A polynomial function is a function of the form:

$$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots a_{n-3}x^3 + a_{n-2}x^2 + a_{n-1}x + a_n$$

where all of the exponents are positive integers and all of the coefficients $a_0 \ldots a_n$ are constants.

As the theory of finding roots of polynomial functions evolved, a 17th century mathematician, Girard (1595-1632) made the following claim which has come to be known as the Fundamental Theorem of Algebra: *An nth degree polynomial function has n roots*.

In later math classes you will study polynomial functions that contain higher-ordered terms such as x^3 or x^5. Based on you work in this task, do you believe this theorem holds for quadratic functions? That is, do all functions of the form $y = ax^2 + bx + c$ always have two roots? [Examine the graphs of each of the quadratic functions you have written equations for in this task. Do they all have two roots? Why or why not?]

3.8 To Be Determined . . . – Teacher Notes

A Develop Understanding Task

Purpose: In the context of using procedures students have developed previously for writing equations for quadratic functions from the information given in a table or a graph, students will examine the nature of the roots of quadratic functions and surface the need for non-real roots when the quadratic function does not intersect the x-axis. This task follows the approach of the historical development of these non-real numbers. As mathematicians developed formulas for solving quadratic and cubic polynomials, the square root of a negative number would sometimes occur in their work. Although such expressions seemed problematic and undefined, when mathematicians persisted in working with these expressions using the same algebraic rules that applied to real-valued radical expressions, the work would lead to correct results. In this task, students will be able to write the equation of quadratic #4 in both vertex and standard form, but attempting to use the quadratic formula to find the roots, and therefore the factored form, will produce expressions that contain the square root of a negative number. However, if students persist in expanding out this factored form using the usual rules of arithmetic, the non-real-valued radical expressions will go away, leaving the same standard form as that obtained by expanding the vertex form. This should give some validity to these non-real-valued radical expressions. It is suggested that these numbers not be referred to as "imaginary" numbers in this task, but only that they are noted to be problematic in the sense of not representing a real value.

(Note: In the early history of mathematics even negative real numbers were considered "fictitious" or "false" solutions to quadratic and cubic equations, although Cardano (1501-1576) and Bombelli (1526-1572) also used square roots of negative numbers in their work. By the 17th century negative numbers were recognized as legitimate solutions to polynomial equations, but complex numbers remained controversial through the 18th century, even though they were useful in the theory of equations. Descartes (1596-1650) called all complex numbers "imaginary" and it was Euler (1707-1783) that introduced the symbol i for the square root of -1. Although expanding the number system to include complex numbers was sufficient to solve quadratic equations, it was not known if complex numbers were sufficient to solve cubic and higher-degree polynomial equations until 1799 when Gauss published a proof that all polynomial equations of degree n have n roots of the form $a + bi$. See a brief history of complex numbers at http://www.clarku.edu/~djoyce/complex/)

Core Standards Focus:

N.CN.7 Solve quadratic equations with real coefficients that have complex solutions.

N.CN.8 Extend polynomial identities to the complex numbers.

N.CN.9 Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

A.REI.4 Solve quadratic equations in one variable.

a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as a ± bi for real numbers a and b.

Note for Mathematics II A.REI.4a, A.REI.4b

Extend to solving any quadratic equation with real coefficients, including those with complex solutions.

Launch (Whole Class):

Explore (Small Group):

Watch for students who start question 1 by locating the x-intercepts of the graph (the zeroes in the table) and then using these values to write the two factors for the factored form. Once the factors are written they might multiply out the expression to get standard form, and then complete the square to get vertex form. This procedural approach works well for question 1, but will not work for questions 2 and 4, since the roots of the quadratic are not readily apparent from the table or the graph. For these students, suggest that Israel and Miriam noticed that all of the parabolas are symmetric about the same line $x = -2$, and so the vertex always lies on this line. Encourage these students to consider how they might use this fact to write the vertex form of each equation.

Watch for how students approach question 3, where the vertex lies on the *x*-axis. It may be difficult for students to recognize that the *x*-intercept at $x = -2$ is a root of multiplicity 2 and that the vertex form is $y = (x+2)^2$. For these students, suggest that Israel and Miriam noticed that as this sequence of parabolas are shifted up along the axis of symmetry the two x-intercepts get closer and closer together until they merge at $x = -2$. This would suggest that we have a "double root" at $x = -2$.

The most useful strategy students might use for question 2 is to write the vertex form by locating the minimum point in either the table or the graph. Once students have written the vertex form they can expand it to get standard form. They can then use the quadratic formula to find the zeros of the function, and then use these zeroes to write the corresponding factors. The irrational roots in question 2 can be found in this way. Have students verify that the radical values found using the quadratic formula fit in the intervals between -4 and -3 and between -1 and 0 by calculating approximate values for these roots using a calculator. In a similar way, students can find the roots of the quadratic in question 4 using the quadratic formula. At this point in time, allow students to write these roots as $-2-\sqrt{-1}$ and $-2+\sqrt{-1}$, you do not need to introduce the notation for complex numbers using *i* to represent the square root of -1. This will be the focus of the next task. Students may try to find approximate values for these roots using a calculator or recognize the dilemma of taking the square root of a negative number as being undefined in terms of real numbers. Acknowledge this dilemma, but also ask students to test their factored form for a few points (see question 5) and to multiply out their factored form in the usual way to verify that it yields the same standard form of the equation that they got when they expanded their vertex form. The goal here is to help students see that these numbers—while undefined in the real number system—yield correct results when manipulated with the familiar rules of algebra.

It is anticipated that students will get bogged down with this algebraic work. You can move to a whole class discussion during problems 4-6 to resolve these algebraic issues.

Discuss (Whole Class):
Begin the whole class discussion by examining question 4. First, have a student present the vertex form of this equation $y=(x+2)^2+1$. Then have a student present the standard form, which can be obtained by multiplying out the vertex form: $y=x^2+4x+5$. Finally ask how we might write the factored form of this function, since it does not cross the *x*-axis and therefore has no *x*-intercepts. If you have identified students who used the quadratic formula to find the "zeros" or "roots" of the quadratic, have them present their work.

Students may have questions about how to write the roots of this quadratic after substituting values for *a, b* and *c* into the quadratic formula, or how to write the factored form since the roots contain two terms. Help support this algebraic work, including simplifying the radical expression $\sqrt{-4}=\sqrt{4}\cdot\sqrt{-1}=2\sqrt{-1}$, and the rational expression $\frac{-4\pm 2\sqrt{-1}}{2}=\frac{-4}{2}\pm\frac{2\sqrt{-1}}{2}=-2\pm\sqrt{-1}$. The factored form is $y=\left[x-\left(-2+\sqrt{-1}\right)\right]\cdot\left[x-\left(-2-\sqrt{-1}\right)\right]=\left(x+2-\sqrt{-1}\right)\left(x+2+\sqrt{-1}\right)$. Once students have written the factored form, ask them to multiply out the two trinomial factors to obtain the standard form. Help them observe that since $\sqrt{-1}\cdot\sqrt{-1}=-1$ is consistent with the properties of radicals we have defined previously, and that this interpretation leads to the same standard form we started with. This consistency of properties will lead us in the next task to define the set of complex numbers.

Once the algebra of working with these negative radical expressions has been demonstrated, have students continue to work on the remainder of the task. Questions 7-9 point out that the algebraic work for irrational roots is similar to the algebraic work for these non-real roots.

Be sure to have a whole class discussion about the Fundamental Theorem of Algebra. Students may not feel like the theorem is true for quadratics, since #3 has only one real root and #4 and #8 have no real roots at all. Point out that we need to count these non-real roots, as well as multiple roots (such as $x = -2$ being a root of multiplicity 2 for question 3) to account for two roots for every quadratic. This will lead to the definition of complex numbers as roots in the next task.

Aligned Ready, Set, Go: Solving Quadratic and Other Equations 3.8

Name: Solving Quadratic and Other Equations | 3.8

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Ready, Set, Go!

Ready

Topic: Simplifying radicals

Simplify each of the radicals below.

1. $\sqrt{8}$
2. $\sqrt{18}$
3. $\sqrt{32}$
4. $\sqrt{20}$
5. $\sqrt{45}$
6. $\sqrt{80}$

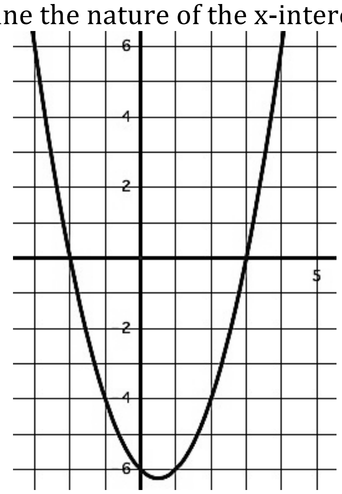
7. What is the connection between the radicals above? Explain.

Set

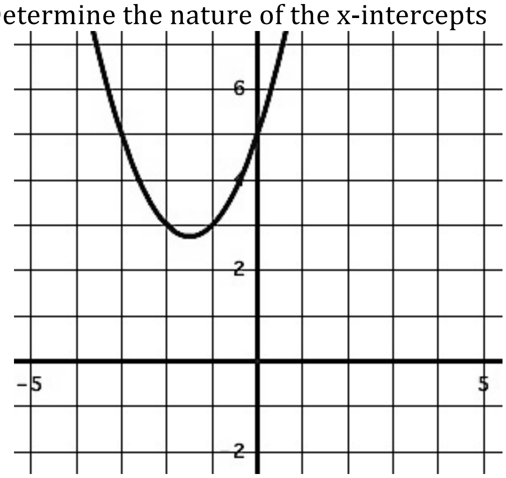
Topic: Determine the nature of the x-intercepts for each quadratic below.

Given the quadratic function, its graph or other information below determine the nature of the x-intercepts (what type of number it is). Explain or show how you know.
(Whole numbers "$\mathbb{W}$", Integers "$\mathbb{Z}$", Rational "$\mathbb{Q}$", Irrational "$\overline{\mathbb{Q}}$", or finally, "not Real")

8. Determine the nature of the x-intercepts.



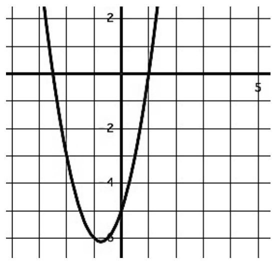
9. Determine the nature of the x-intercepts



10. Determine the nature of the x-intercepts.
$f(x) = x^2 + 4x - 24$

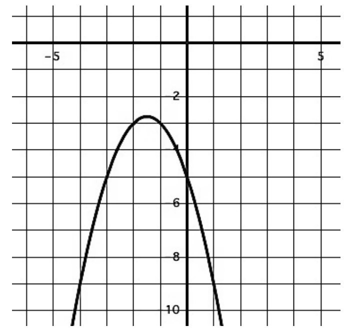
11. Determine the nature of the x-intercepts.
$g(x) = (2x - 1)(5x + 2)$

12. Determine the nature of the x-intercepts.



$f(x) = 2x^2 + 3x - 5$

13. Determine the nature of the x-intercepts.



14. Determine the nature of the x-intercepts.
$r(t) = t^2 - 8t + 16$

15. Determine the nature of the x-intercepts.
$h(x) = 3x^2 - 5x + 9$

Determine the number of roots that each polynomial will have.

16. $x^5 + 7x^3 - x^2 + 4x - 21$ 17. $4x^3 + 2x^2 - 3x - 9$ 18. $2x^7 + 4x^5 - 5x^2 + 16x + 3$

Go

Topic: Finding x-intercepts for quadratics using factoring and quadratic formula.

If the given quadratic function can be factored then factor and provide the x-intercepts. If you cannot factor the function then use the quadratic formula to find the x-intercepts.

19. $A(x) = x^2 + 4x - 21$ 20. $B(x) = 5x^2 + 16x + 3$ 21. $C(x) = x^2 - 4x + 1$

22. $D(x) = x^2 - 16x + 4$ 23. $E(x) = x^2 + 3x - 40$ 24. $F(x) = 2x^2 - 3x - 9$

25. $G(x) = x^2 - 3x$ 26. $H(x) = x^2 + 6x + 8$ 27. $K(x) = 3x^2 - 11$

Need Assistance? Check out these additional resources:

https://www.khanacademy.org/math/algebra/quadratics/quadratic_formula/v/quadratic-formula-1

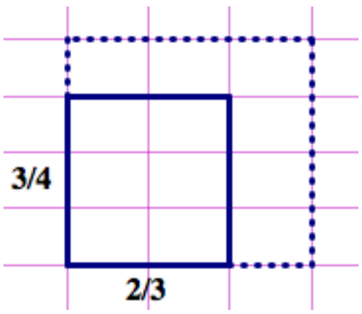
3.9 My Irrational and Imaginary Friends

A Solidify Understanding Task

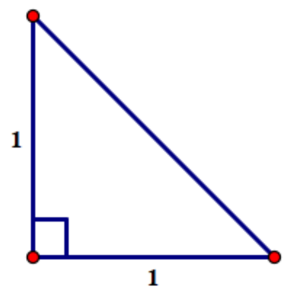
Part 1: Irrational numbers

Find the perimeter of each of the following figures. Express your answer as simply as possible.

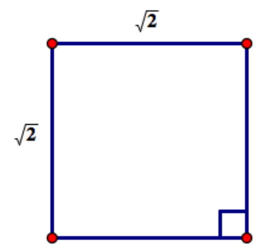
1. The 3/4 x 2/3 rectangle



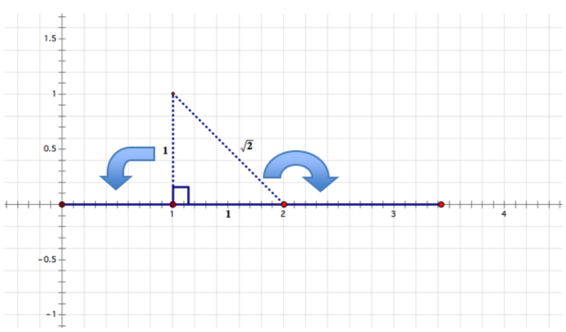
2. The isosceles right triangle



3. The √2 x √2 square



4. We might approximate the perimeter of figure 2 with a decimal number, but the exact perimeter is $2+\sqrt{2}$, which cannot be simplified any farther. Note that this notation represents a single number—the distance around the perimeter of the triangle—even though it is written as the sum of two terms. We could visualize this single number by laying the three sides of the triangle end-to-end along a number line, starting at 0, so the endpoint of the last segment would be at the number $2+\sqrt{2}$. Is the number we have located on the number line in this way a rational number or an irrational number? Explain your answer.



5. Why can we express the perimeter of figure 3 with a single term, even though the side lengths are irrational?

6. Find the area of each of the figures in 1-3.

7. Draw a representative image and find the area of the following figures:

(a) a square with sides $2+\sqrt{2}$

(b) a rectangle with sides $3+\sqrt{2}$ and $2+\sqrt{5}$

(c) a rectangle with sides $3+\sqrt{2}$ and $2+\sqrt{8}$

(d) a rectangle with sides $\sqrt{2}+\sqrt[3]{2}$ and $\sqrt{6}+\sqrt[3]{4}$

8. Are the areas of the figures in 7a, 7b, 7c and 7d rational or irrational? How do you know?

Note: The set of numbers that contains all of the *rational numbers* and all of the *irrational numbers* is called the set of *real numbers*.

Part 2: Imaginary and Complex Numbers

In the previous task, you found that the quadratic formula gives the roots of $x^2 + 4x + 5$ as $-2 + \sqrt{-1}$ and $-2 - \sqrt{-1}$. Because the square root of a negative number has no defined value as either a rational or an irrational number, Euler proposed that a new number $i = \sqrt{-1}$ be including in what came to be known as the complex number system.

9. Based on Euler's definition of *i*, what would the value of i^2 be?

With the introduction of the number *i*, the square root of *any* negative number can be represented. For example, $\sqrt{-2} = \sqrt{2} \cdot \sqrt{-1} = \sqrt{2} \cdot i$ and $\sqrt{-9} = \sqrt{9} \cdot \sqrt{-1} = 3i$.

10. Find the values of the following expressions. Show the details of your work.

   (a) $\left(\sqrt{2} \cdot i\right)^2$

   (b) $3i \times 3i$

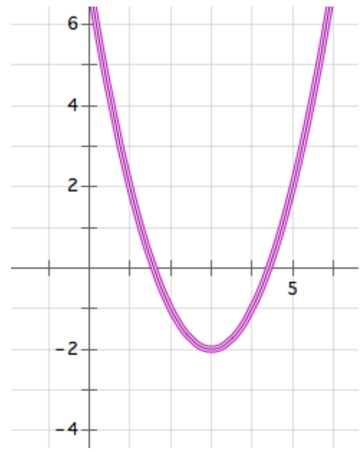
Using this new notation, the roots of $x^2 + 4x + 5$ can be written as $-2 + i$ and $-2 - i$, and the factored form of $x^2 + 4x + 5$ can be written as $(x + 2 - i)(x + 2 + i)$.

11. Verify that $x^2 + 4x + 5$ and $(x + 2 - i)(x + 2 + i)$ are equivalent by expanding and simplifying the factored form. Show the details of your work.

Note: Numbers like 3*i* and $\sqrt{2} \cdot i$ are called *pure imaginary numbers*. Numbers like -2 – *i* and -2 + *i* that include a real term and an imaginary term are called *complex numbers*.

The quadratic formula is usually written in the form $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. An equivalent form is $\frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$. If a, b and c are rational coefficients, then $\frac{-b}{2a}$ is a rational term, and $\frac{\sqrt{b^2 - 4ac}}{2a}$ may be a rational term, an irrational term or an imaginary term, depending on the value of the expression under the square root sign.

12. Examine the roots of the quadratic $y = x^2 - 6x + 7$ shown in the graph at the right. How do the terms $\frac{-b}{2a}$ and $\frac{\sqrt{b^2 - 4ac}}{2a}$ show up in this graph?

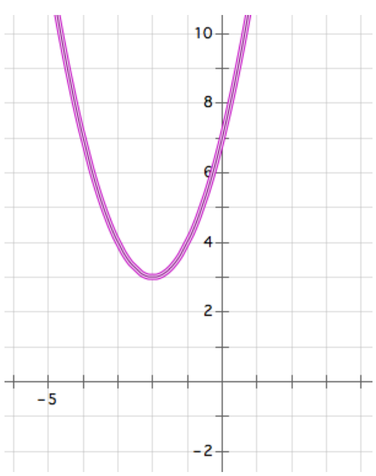


Look back at the work you did in the task *To Be Determined . . .*

13. Which quadratics in that task had complex roots?

14. How can you determine if a quadratic has complex roots from its graph?

15. Find the complex roots of the following quadratic function represented by its graph.



Note: Complex numbers are not real numbers—they do not lie on the real number line that includes all of the rational and irrational numbers; also note that the real numbers are a subset of the complex numbers since a real number results when the imaginary part of *a* + *bi* is 0, that is, *a* + 0*i*.

The Fundamental Theorem of Algebra, Revisited

Remember the following information given in the previous task:

A polynomial function is a function of the form:

$$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots a_{n-3}x^3 + a_{n-2}x^2 + a_{n-1}x + a_n$$

where all of the exponents are positive integers and all of the coefficients $a_0 \ldots a_n$ are constants.

As the theory of finding roots of polynomial functions evolved, a 17th century mathematician, Girard (1595-1632) made the following claim which has come to be known as the Fundamental Theorem of Algebra: *An nth degree polynomial function has n roots*.

Based on you work in this task, do you believe this theorem holds for quadratic functions? That is, do all functions of the form $y = ax^2 + bx + c$ always have two roots?

3.9 My Irrational and Imaginary Friends – Teacher Notes

A Solidify Understanding Task

Purpose: The purpose of this task is to practice the arithmetic of irrational numbers, mainly non-rational radical numbers, as well as the arithmetic of complex numbers. Students will note similarities and differences in the rules of simplifying such expressions.

Unlike rational numbers, where we can always find a common rational unit of measure that fits into two different rational lengths, irrational numbers are said to be incommensurate, since no rational unit of measure can be found that fits into an irrational length evenly. For example, a length of $\frac{2}{3}+\frac{3}{4}$ can be accurately measured using a $\frac{1}{12}$ unit of length. On the other hand, a length of $\pi+\sqrt{2}$ cannot be measured with any rational unit of length. For example, decimal units—tenths, hundredths, thousandths, etc.—cannot be used to measure an irrational length exactly. Consequently, there is no exact decimal representation for irrational lengths. We can only approximate irrational lengths with a finite decimal number. Thinking about rational and irrational number from a geometric perspective will give students a sense of what it means to add, subtract and multiply when one addend or factor is rational and the other is irrational. This discussion is then extended to consider what happens when a real number is added to an imaginary or complex number.

Core Standards Focus:

N.RN.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Note for Mathematics II: *Connect N.RN.3 to physical situations, e.g., finding the perimeter of a square of area 2.*

N.CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form a + bi with a and b real.

N.CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Note for Mathematics II*: Limit to multiplications that involve i^2 as the highest power of i.*

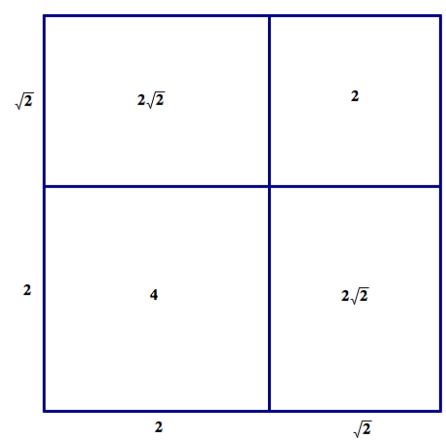
N.CN.7 Solve quadratic equations with real coefficients that have complex solutions.

N.CN.8 Extend polynomial identities to the complex numbers.

N.CN.9 Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Launch (Whole Class): [part 1: Irrational numbers]

Launch this task by having students work on the arithmetic for finding the perimeters of the figures in problems 1, 2 and 3, and then discuss their results (see discussion questions 4-5). In order to find the perimeter of figure 1 students will find a common unit of measure for all of the sides, $\frac{1}{12}$, and then determined how many of those units fit around the perimeter. However, they may not be aware that the arithmetic they are doing can be thought about in this way. Help them explain their work for problem 1 in terms of the size of the unit of measure they used and the number of those units in the perimeter. Illustrate this by marking the sides in smaller units of $1/12$. Students' answer for problem 2 will be $2+\sqrt{2}$. Point out that in problem 1, students were able to combine the lengths of all of the sides into a single term by finding a unit fraction that could be used as a common unit to measure all of the sides. Pose the question, "Do you think there is a small unit fraction that could be used to divide up each of the sides of the right triangle in problem 2 so that we could combine all of the lengths of the sides of the triangle into a single term, like we did for problem 1?" Students may initially believe this might be possible, perhaps by using a very small unit fraction like $1/100$ or $1/1000$, but an examination of a decimal approximation of $\sqrt{2}$ would suggest we would have to go out to several decimal places to find such a unit. The ancient Greeks, prior to Hippasus and Zeno, believed that all lengths were discrete and composed of a finite number of units of a given size. Hippasus proved that there was no common unit of measure for the right triangle given in question 2 using a proof by contradiction: If one assumes there is a common unit of measure for both the hypotenuse and the legs of the right triangle, then it can be shown that a leg must contain both an even and an odd number of those units; this contradiction implies that the assumption that there is such a unit is false. The details of such a proof can be found at http://en.wikipedia.org/wiki/Irrational_number. You may choose to share the details of such a proof with your students, or just share a description of the work and the results, as we have done here. The goal of this discussion is to point out that the sum of the lengths of the sides (i.e., the perimeter) of the right triangle in question 2 has to be written as a sum of two terms—a rational term and an irrational term—since these two terms are "incommensurable magnitudes." On the other hand, the perimeter of the square in problem 3 can be written as a single term, $4\sqrt{2}$, since the irrational length $\sqrt{2}$ is common to all four sides. However, the product of the rational number 4 and the irrational number $\sqrt{2}$ is an irrational number, since the total length is just 4 times the original irrational length.



Assign students to work on questions 6-8 with a partner. On question 7 students should use an "area model" representation to decompose the area of the rectangles into smaller pieces. For example, a square with side lengths $2+\sqrt{2}$ can be decomposed as shown in the diagram. The area is represented by the number $6+4\sqrt{2}$, an irrational number.

Explore (Small Group): [part 1: Irrational numbers]
As students create their area models they will notice that the product of two irrational numbers is sometimes a rational number (e.g., $\sqrt{2}\cdot\sqrt{8}=\sqrt{16}=4$) and sometimes an irrational number (e.g., $\sqrt{2}\cdot\sqrt{5}=\sqrt{10}$). As you observe students, make note of any algebraic work that may need to be discussed as a whole class.

Discuss (Whole Class): [part 1: Irrational numbers]
Clarify any algebraic work that seemed difficult or problematic for students. Many issues can be resolved by rewriting the radical expressions with exponents, and then using the properties of exponents to simplify the expression.

Discuss question 8, "How do you know if the areas of the figures are rational or irrational?" For square roots, a product of two irrational numbers is rational only if the product of the radicands can be decomposed into a product of perfect squares, or the prime factorization of the product of the radicands contains only prime factors raised to even powers. Ask students to state a corresponding rule for cube roots.

Launch (Whole Class): [part 2: Imaginary and complex numbers]
Introduce Euler's notation for representing $\sqrt{-1}$ with *i*, and then discuss question 9 and the example that precedes question 10. Assign students to work on the rest of the task.

Explore (Small Group): [part 2: Imaginary and complex numbers]
Monitor students' work on simplifying sums and products of complex numbers and identify any algebraic work that needs to be discussed as a whole class.

Discuss (Whole Class): [part 2: Imaginary and complex numbers]
Clarify any algebraic work that seemed difficult or problematic for students. Discuss how one can tell if the roots of a quadratic function are complex from a graph. Have students share how they found the complex roots for the quadratic whose graph is shown in question 15. One strategy would be to write the vertex form of the quadratic function and set it equal to 0 and then solve the resulting quadratic equation, either by isolating the squared-binomial and taking the square root of both sides of the equation, or by expanding out the binomial to get standard form and then using the quadratic function.

Revisit the discussion about the Fundamental Theorem of Algebra from the previous task, and point out that a quadratic function has two roots when real, complex and multiple roots are considered.

Aligned Ready, Set, Go: Solving Quadratic and Other Equations 3.9

Name: Solving Quadratic and Other Equations | 3.9

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Ready, Set, Go!

Ready

Topic: Classifying numbers according to set.

Classify each of the numbers represented below according to the sets to which they belong. If a number fits in more than one set then list all that apply.
(Whole numbers "$\mathbb{W}$", Integers "$\mathbb{Z}$", Rational "$\mathbb{Q}$", Irrational "$\overline{\mathbb{Q}}$", Real "$\mathbb{R}$", Complex "$\mathbb{C}$")

1. π
2. -13
3. $\sqrt{-16}$
4. 0
5. $\sqrt{75}$
6. $\frac{9}{3}$
7. $\sqrt{\frac{4}{9}}$
8. $5 + \sqrt{2}$
9. $\sqrt{-40}$

Set

Topic: Simplifying radicals, imaginary numbers

Simplify each radical expression below.

10. $3 + \sqrt{2} - 7 + 3\sqrt{2}$
11. $\sqrt{5} - 9 + 8\sqrt{5} + 11 - \sqrt{5}$
12. $\sqrt{12} + \sqrt{48}$
13. $\sqrt{8} - \sqrt{18} + \sqrt{32}$
14. $11\sqrt{7} - 5\sqrt{7}$
15. $7\sqrt{7} + 5\sqrt{3} - 3\sqrt{7} + \sqrt{3}$

Simplify. Express as a complex number using " *i* " if necessary.

16. $\sqrt{-2} \cdot \sqrt{-2}$
17. $7 + \sqrt{-25}$
18. $(4i)^2$
19. $i^2 \cdot i^3 \cdot i^4$
20. $\left(\sqrt{-4}\right)^3$
21. $(2i)(5i)^2$

Solve each quadratic equation over the set of complex numbers.

22. $x^2 + 100 = 0$

23. $t^2 + 24 = 0$

24. $x^2 - 6x + 13 = 0$

25. $r^2 - 2r + 5 = 0$

Go

Topic: Solve quadratic equations.

Use the discriminant to determine the nature of the roots to the quadratic equation.

26. $x^2 - 5x + 7 = 0$

27. $x^2 - 5x + 6 = 0$

28. $2x^2 - 5x + 5 = 0$

29. $x^2 + 7x + 2 = 0$

30. $2x^2 + 7x + 6 = 0$

31. $2x^2 + 7x + 7 = 0$

32. $2x^2 - 7x + 6 = 0$

33. $2x^2 + 7x - 6 = 0$

34. $x^2 + 6x + 9 = 0$

Solve the quadratic equations below using an appropriate method.

35. $m^2 + 15m + 56 = 0$

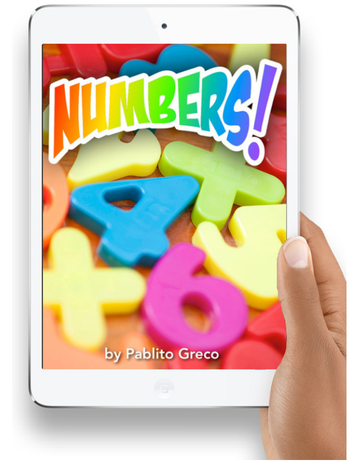
36. $5x^2 - 3x + 7 = 0$

37. $x^2 - 10x + 21 = 0$

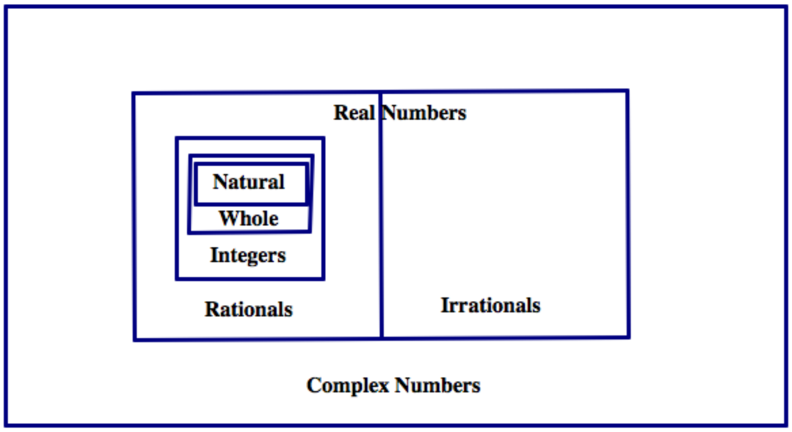
38. $6x^2 + 7x - 5 = 0$

3.10 iNumbers

A Practice Understanding Task



In order to find solutions to all quadratic equations, we have had to extend the number system to include complex numbers.



Do the following for each of the problems below:

- Choose the best word to complete each conjecture.
- After you have made a conjecture, create at least four examples to show why your conjecture is true.
- If you find a counter-example, change your conjecture to fit your work.

Conjecture #1: The sum of two integers is [always, sometime, never] an integer.

Conjecture #2: The sum of two rational numbers is [always, sometimes, never] a rational number.

Conjecture #3: The sum of two irrational numbers is [always, sometimes, never] an irrational number.

Conjecture #4: The sum of two real numbers is [always, sometimes, never] a real number.

Conjecture #5: The sum of two complex numbers is [always, sometimes, never] a complex number.

Conjecture #6: The product of two integers is [always, sometime, never] an integer.

Conjecture #7: The quotient of two integers is [always, sometime, never] an integer.

Conjecture #8: The product of two rational numbers is [always, sometimes, never] a rational number.

Conjecture #9: The quotient of two rational numbers is [always, sometimes, never] a rational number

Conjecture #10: The product of two irrational numbers is [always, sometimes, never] an irrational number.

Conjecture #11: The product of two real numbers is [always, sometimes, never] a real number.

Conjecture #12: The product of two complex numbers is [always, sometimes, never] a complex number.

13. The ratio of the circumference of a circle to its diameter is given by the irrational number π. Can the diameter of a circle and the circumference of the same circle both be rational numbers? Explain why or why not.

The Arithmetic of Polynomials

In the task *To Be Determined . . .* we defined polynomials to be expressions of the following form:

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots a_{n-3}x^3 + a_{n-2}x^2 + a_{n-1}x + a_n$$

where all of the exponents are positive integers and all of the coefficients $a_0 \ldots a_n$ are constants.

Do the following for each of the problems below:

- Choose the best word to complete each conjecture.
- After you have made a conjecture, create at least four examples to show why your conjecture is true.
- If you find a counter-example, change your conjecture to fit your work.

Conjecture #P1: The sum of two polynomials is [always, sometime, never] a polynomial.

Conjecture #P2: The difference of two polynomials is [always, sometime, never] a polynomial.

Conjecture #P3: The product of two polynomials is [always, sometime, never] a polynomial.

3.10 iNumbers – Teacher Notes

A Practice Understanding Task

Purpose: The purpose of this task is to practice working with the arithmetic of irrational and complex numbers and to make conjectures as to which of the sets of integers, rational numbers, irrational numbers, real numbers or complex numbers are closed under the operations of addition, subtraction and multiplication; that is, the sum or product of any two numbers from the set always produces another number in that set. Students also experiment with the closure of the set of polynomial functions under the operations of addition, subtraction and multiplication. From this perspective, the set of integers behave in the same way as the set of polynomials. Once polynomial division has been introduced in Mathematics III, it can be shown that neither set is closed under the operation of division—dividing two integers results in a rational number and dividing two polynomials results in a rational function.

Core Standards Focus:

N.RN.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Note for Mathematics II: *Connect N.RN.3 to physical situations, e.g., finding the perimeter of a square of area 2.*

N.CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form a + bi with a and b real.

N.CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Note for Mathematics II: *Limit to multiplications that involve i^2 as the highest power of i.*

A.APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Note for Mathematics II: *Focus on polynomial expressions that simplify to forms that are linear or quadratic in a positive integer power of x.*

Related Standards:

Launch (Whole Class):
Examine the Venn diagram of the complex number system given at the beginning of the task. Ask students to suggest a quadratic equation that would not have a solution if we limited the set of acceptable solutions to only natural numbers (for example, $x^2 + 3x = 0$ would not be solvable). What if we limited the set of acceptable solutions to only integers? (Then $(2x+1)(3x-2) = 0$ would not be solvable.) What if we limited the set of acceptable solutions to rational numbers? Real numbers? Point out to students that we have expanded our number system to allow us to solve a greater variety of equations.

Let students know that their work today is to make conjectures about operations within these sets of numbers. Can we always do arithmetic within in a set of numbers without having to go outside the set to get an answer? Mathematicians refer to this property of a set of numbers as *closure*.

Explore (Small Group):
As students explore the conjectures, make sure they are trying out a range of possibilities, and trying to search for counter-examples to the "always" conjectures. Remind students of the work of the previous task as they consider operations with irrational and complex numbers.

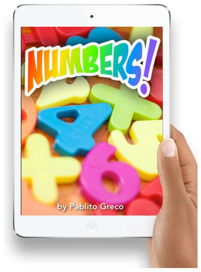
Monitor students' work with the arithmetic of polynomials: Are they writing only polynomial expressions in their examples? How do they add two polynomial expressions? Do they only add like terms? How do they multiply polynomial expressions that contain multiple terms? Do they correctly use the distributive property? How do they multiply variables raised to powers?

Discuss (Whole Class):
As needed, share conjectures and supporting evidence with the whole class. Discuss the question about the definition of $\pi = \frac{C}{d}$. Since π is an irrational number, this ratio implies that the circumference and diameter of a circle cannot both be rational numbers. Including π in the discussion of irrational numbers allows you to point out that there is more to the set of irrational numbers than just radicals.

Also discuss the algebraic work with polynomial expressions to resolve any misconceptions you may have observed during this work.

Aligned Ready, Set, Go: Solving Quadratic and Other Equations 3.10

Ready, Set, Go!



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Ready

Topic: Attributes of quadratics and other functions

1. Summarize what you have learned about quadratic functions to this point. In addition to your written explanation provide graphs, tables and examples to illustrate what you know.

2. In prior work you have learned a great deal about both linear and exponential functions. Compare and contrast linear and exponential functions with quadratic functions. What similarities if any are there and what differences are there between linear, exponential and quadratic functions?

Set

Topic: Operations on different types of numbers

3. The Natural numbers, $\mathbb{N}$, are just that the numbers that come naturally or the counting numbers. As any child first learns numbers they learn 1, 2, 3, … What operations on the Natural numbers would cause the need for other types of numbers? What operation on Natural numbers create a need for Integers or Rational numbers and so forth. (Give examples and explain.)

In each of the problems below use the given items to determine whether or not it is possible *always*, *sometimes* or *never* to create a new element* **that is in the desired set.**

4. Using the operation of addition and elements from the Integers, $\mathbb{Z}$, [always, sometime, never] an element of the Irrational numbers, $\overline{\mathbb{Q}}$, will be created. Explain.

5. Consider the equation $a - b = c$, where $a \in \mathbb{N}$ and $b \in \mathbb{N}$, c will be an Integer, $\mathbb{Z}$ [always, sometimes, never]. Explain.

6. Consider the equation $a \div b = c$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, then is $c \in \mathbb{Z}$ [sometimes, always, never]. Explain.

**The numbers in any given set of numbers may be referred to as elements of the set. For example, the Rational number set, $\mathbb{Q}$, contains elements or numbers that can be written in the form $\frac{a}{b}$, where a and b are integer values (b≠0).*

7. Using the operation of subtraction and elements from the Irrationals, $\overline{\mathbb{Q}}$, an element of the Irrational numbers, $\overline{\mathbb{Q}}$, will be created [always, sometime, never]. Explain.

8. If two Complex numbers, $\mathbb{C}$, are subtracted the result will [always, sometimes, never] be a Complex number, $\mathbb{C}$. Explain.

Go

Topic: Solving all types of Quadratic Equations, Simplifying Radicals

Make a prediction as to the nature **of the solutions for each quadratic (Real, Complex, Integer, etc.) then solve each of the quadratic equations below using an appropriate and efficient method. Give the solutions and compare to your prediction.**

9. $-5x^2 + 3x + 2 = 0$

Prediction:

Solutions:

10. $x^2 + 3x + 2 = 0$

Prediction:

Solutions:

11. $x^2 + 3x - 12 = 0$

Prediction:

Solutions:

12. $4x^2 - 19x - 5 = 0$

Prediction:

Solutions:

Simplify each of the radical expressions. Use rational exponents if desired.

13. $\sqrt[4]{81x^8y^{12}}$

14. $\sqrt{\frac{a^7b^{10}}{a^3}}$

15. $\sqrt[5]{625x^{12}}$

16. $(\sqrt{n})^5$

17. $\sqrt[3]{-27}$

18. $(\sqrt{8})(\sqrt{3^2})(2)$

Fill in the table so each expression is written in radical form and with rational exponents.

	Radical Form	Exponential Form
19.	$\sqrt[4]{8^3}$	
20.		$256^{\frac{3}{4}}$
21.	$\sqrt[4]{2^7 \bullet 4^5}$	
22.		$16^{\frac{3}{2}} \bullet 4^{\frac{1}{2}}$
23.	$\sqrt[10]{x^{23}y^{31}}$	
24.	$\sqrt[5]{64a^9b^{18}}$	